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Weighted Statistical Approximation by Gamma Type operators *

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Abstract
In this paper, with the help of Korovkin type theorem, we study the weighted statistical approximation properties of a kind of Gamma type operators which preserve $e^{2\mu x}$ $(\mu > 0)$. Further, the rate of statistical convergence is given.

Keywords: Gamma type operators; Korovkin type theorem; weighted statistical approximation.

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1. INTRODUCTION

The famous Gamma operator, which was introduced by Lupas and Müller[1], is given by

$$G_n(f;x) = \frac{1}{n!}\int_0^\infty e^{-\tau}\tau^n f(\frac{nx}{\tau})d\tau, \qquad x \in (0,\infty).$$

The Gamma operators were studied extensively[1−5]. Draganov and Ivanov[2] gave a brief summary of the results related to the rate of global convergence in terms of weighted K-functionals and contained in [3,4,7]. In order to improve the approximation effect, Deveci, Acar and Alagoz[8] introduced a refinement of Gamma operators which preserve constants and $e^{2\mu}$ $(\mu > 0)$ functions. The concept of statistical convergence, which was first introduced by Fast[9] in 1951, is a generalization of the ordinary convergence. Several extensions of statistical approximation processes have appeared in literature [10-15] and references therein.

In this paper, we investigate the statistical approximation properties of the operators $G_{n,\mu}(f;x)$ which preserve 1 and $e^{2\mu x}$ $(\mu > 0)$.

2. THE PROPERTIES OF THE OPERATORS

Let us consider the following operators for each positive integer n and $\mu > 0$[8]:

$$G_{n,\mu}(f;x) = \frac{1}{n!}\int_0^\infty e^{-\tau}\tau^n f(\frac{x^2\tau}{n\alpha})d\tau, \quad (x > 0)$$

where

$$\alpha = \frac{2e^{\frac{2x\mu}{n+1}}\mu x^2}{n(e^{\frac{2\mu x}{n+1}} - 1)}.$$

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Now, we recall the following results of the operators, the details can be found in [8].

Lemma 2.1.[8, Lemma 1] For $\mu > 0$, $x \in (0,\infty)$, then

$$G_{n,\mu}(1;x) = 1;$$

$$G_{n,\mu}(e^{\mu t};x) = (1 + \frac{\mu x^2}{n\alpha - \mu x^2})^{n+1};$$

$$G_{n,\mu}(e^{2\mu t};x) = e^{2\mu x}.$$

Lemma 2.2.[8, Lemma 3] For $\lambda \in (-\infty,+\infty)$, $\mu > 0$, $x \in (0,\infty)$, one has

$$\lim_{n\to+\infty} G_{n,\mu}(e^{-\lambda t};x) = e^{-\lambda x}.$$

Remark 1. $C^*(0,\infty) := \{f \in C(0,\infty) : \lim_{x\to+\infty} f(x) \text{ exists and is finite}\}$. The space of such functions is endowed with the uniform norm $\| f \|_\infty := \sup_{x\in(0,\infty)} |f(x)|$.

Remark 2.[8, Theorem 2] Let $\mu > 0$, for the sequence of operators $G_{n,\mu} : C^*(0,\infty) \to C^*(0,\infty)$, the convergence $G_{n,\mu}(f;x) \to f(x)$ as $n \to \infty$ is uniformly in $(0,\infty)$, for all $f \in C^*(0,\infty)$.

3. NOTATIONS OF STATISTICAL CONVERGENCE

The following definitions, notations can be found in [9-12].

Definition 3.1.[10] Suppose that $E \subseteq N = \{1,2,3,\cdots\}$, $E_n = \{k \le n : k \in E\}$. The natural density of E is denoted by

$$\delta(E) = \lim_{n\to\infty}\frac{|E_n|}{n},$$

here $|E_n|$ denotes the cardinality of the enclosed set E_n. A sequence $x = (x_k)$ is said to be statistically convergent to L, if, for every $\varepsilon > 0$, $\delta(\{k \in N : |x_k - L| \ge \varepsilon\}) = 0$. In symbol, we write $S - \lim x = L$ or $S - \lim_n x_n = L$.

Remark 3. Every convergent sequence is statistically convergent, but its converse is not always valid.

Definition 3.2.[15] A given non-negative infinite summability matrix $A = (a_{n,k})$ is said to be regular if $\lim_n(Ax)_n = \lim_n \sum_{k=1}^\infty a_{n,k}x_k = L$ whenever $\lim_k x_k = L$. Then the sequence $x = (x_k)$ is said to be A-statistically convergent to L, denoted by $S_A - \lim_n x_n = L$ or $S_A - \lim x = L$, provided that for each $\varepsilon > 0$, $\lim_n \sum_{k:|x_k-L|\ge\varepsilon} a_{n,k} = 0$.

Definition 3.3.[15] Let $p = (p_k)$ be a sequence of nonnegative numbers such that $p_0 > 0$ and $P_n = \Sigma_{k=0}^n p_k \to \infty$ as $n \to \infty$. Matrix $A = (a_{n,k})$ is non-negative infinite regular summability. Then $x = (x_k)$ is said to be weighted A-statistically convergent to L, if, for any $\varepsilon > 0$, $\lim_n \frac{1}{P_n}\Sigma_{k=0}^n p_k \sum_{m:p_m|x_m-L|\geq\varepsilon} a_{k,m} = 0$. In this case, we write $S_A^{\bar{N}} - \lim_n x_n = L$ or $S_A^{\bar{N}} - \lim x = L$.

Definition 3.4.[15] Let $A = (a_{n,k})$ be a non-negative regular summability matrix, $p = (p_k)$ be a sequence of non-negative numbers such that $p_0 > 0$ and $P_n = \sum_{k=0}^n p_k \to \infty$ as $n \to \infty$, and (u_n) be a positive non-increasing sequence. Then $x = (x_k)$ is weighted A-statistically convergent to L with the rate $o(u_n)$, if for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{u_n P_n} \sum_{k=0}^{n} p_k \sum_{m:p_m|x_m-L|\geq\varepsilon} a_{k,m} = 0.$$

This relation is denoted by $S_A^{\bar{N}} - o(u_n) - \lim_n x_n = L$.

4. SOME STATISTICAL APPROXIMATION THEOREMS

In this section, we estimate the properties of the weighted A-statistical convergence of the operators $G_{n,\mu}(f;x)$.

Theorem 4.1. Let $A = (a_{n,k})$ be a non-negative regular summability matrix. For $n \in N$ and $f \in C^*(0,\infty)$, one has

$$S_A^{\bar{N}} - \lim_{n\to\infty} \|G_{n,\mu}(f;x) - f(x)\|_\infty = 0,$$

if and only if

$$S_A^{\bar{N}} - \lim_{n\to\infty} \|G_{n,\mu}(1;x) - 1\|_\infty = 0,$$

$$S_A^{\bar{N}} - \lim_{n\to\infty} \|G_{n,\mu}(e^{-t};x) - e^{-x}\|_\infty = 0,$$

$$S_A^{\bar{N}} - \lim_{n\to\infty} \|G_{n,\mu}(e^{-2t};x) - e^{-2x}\|_\infty = 0.$$

Proof. We only need to prove the sufficient conditions. For $f \in C^*(0,\infty)$, there is a constant $C > 0$, such that $|f(x)| \leq C$. Therefore, $|f(t) - f(x)| \leq 2C$, $0 < t, x < +\infty$. For any $\varepsilon > 0$, there is a $\delta(\varepsilon) > 0$ such that $|f(t) - f(x)| < \varepsilon$, $\forall |e^{-t} - e^{-x}| < \delta$. Consider $D(\delta)$ of the form $D(\delta) = \{(x,t) \in (0,\infty) : |e^{-t} - e^{-x}| < \delta\}$, we obtain

$$\begin{aligned} |f(t) - f(x)| &\leq |f(t) - f(x)|_{D(\delta)} + |f(t) - f(x)|_{(0,\infty)-D(\delta)} \\ &\leq \varepsilon + 2C\frac{(e^{-t} - e^{-x})^2}{\delta^2} \\ &= \varepsilon + \frac{2C}{\delta^2}\Omega, \end{aligned}$$

where $\Omega = (e^{-t} - e^{-x})^2$. For $m \in N$, by a direct computation, we write

$$G_{m,\mu}(\Omega;x) = [G_{m,\mu}(e^{-2t};x) - e^{-2x}]$$

$$-2e^{-x}[G_{m,\mu}(e^{-t};x) - e^{-x}] + e^{-2x}[G_{m,\mu}(1;x) - 1].$$

The term $G_{m,\mu}(f;x) - f(x)$ can be written as

$$\begin{aligned} & |G_{m,\mu}(f;x) - f(x)| \leq \varepsilon G_{m,\mu}(1;x) \\ + \quad & \frac{2C}{\delta^2} G_{m,\mu}(\Omega;x) + |f(x)(G_{m,\mu}(1;x) - 1)| \\ \leq \quad & \varepsilon + \frac{2C}{\delta^2} \parallel G_{m,\mu}(e^{-2t};x) - e^{-2x} \parallel_\infty \\ + \quad & \frac{4C}{\delta^2} \parallel G_{m,\mu}(e^{-t};x) - e^{-x} \parallel_\infty \\ \leq \quad & \frac{4C}{\delta^2} (\parallel G_{m,\mu}(e^{-2t};x) \\ - \quad & e^{-2x} \parallel_\infty + \parallel G_{m,\mu}(e^{-t};x) - e^{-x} \parallel_\infty), \end{aligned}$$

For a given $\varepsilon' > 0$, such that $0 < \varepsilon < \varepsilon'$. If we define the following sets:

$$E = \{m \in N : p_m|G_{m,\mu}(f;x) - f(x)| \geq \varepsilon'\};$$

$$E_1 = \{m \in N : p_m|G_{m,\mu}(e^{-t};x) - e^{-x}| \geq \frac{\varepsilon' - \varepsilon}{8C}\delta^2\};$$

$$E_2 = \{m \in N : p_m|G_{m,\mu}(e^{-2t};x) - e^{-2x}| \geq \frac{\varepsilon' - \varepsilon}{8C}\delta^2\},$$

we see that $E \subset E_1 \cup E_2$,

$$\frac{1}{P_n}\sum_{k=0}^{n} p_k \sum_{m\in E} a_{k,m} \leq \frac{1}{P_n}\sum_{k=0}^{n} p_k \sum_{m\in E_1\cup E_2} a_{k,m}.$$

Taking the limit $n \to \infty$ and noting the conditions, we obtain

$$S_A^{\bar{N}} - \lim_{n\to\infty} \parallel G_{n,\mu}(f;x) - f(x) \parallel_\infty = 0.$$

Remark 4. Here we use the Korovkin test functions $\{1, e^{-x}, e^{-2x}\}$. We can also use the usual test functions $\{1, x, x^2\}$.

Theorem 4.2. Let $A = (a_{n,k})$ be a non-negative regular summability matrix. If the following condition yields:

$$S_A^{\bar{N}} - o(u_n) - \lim_n \omega(f;h_n) = 0 \quad on \quad (0,\infty),$$

where $\omega(f;\delta)$ is the classical modulus of continuity which is defined by[3]

$$\omega(f;\delta) = \sup_{x,t>0,|t-x|\leq\delta} |f(x) - f(t)|.$$

Let $h_n = \parallel G_{n,\mu}((t-x)^2;x) \parallel_\infty^{\frac{1}{2}}$, then for $f \in C_B(0,\infty) := \{f \in C(0,\infty) : f$ are bounded and continuous functions$\}$, we have

$$S_A^{\bar{N}} - o(u_n) - \lim_n \parallel G_{n,\mu}(f;x) - f(x) \parallel_\infty = 0.$$

Proof. For $f \in C_B(0,\infty), m \in N$, one has

$$\begin{aligned} |G_{m,\mu}(f;x) - f(x)| &\leq \omega(f;\xi)|G_{m,\mu}(\tfrac{|t-x|}{\xi} + 1;x)| \\ &\leq \omega(f;\xi) + \omega(f;\xi)\tfrac{1}{\xi}|G_{m,\mu}((t-x)^2;x)|^{\frac{1}{2}}. \end{aligned}$$

Let $\xi := h_m$, taking supermum over $x \in (0,\infty)$ on both sides, we obtain

$$\begin{aligned} & \parallel G_{m,\mu}(f;x) - f(x) \parallel_\infty \leq \omega(f;h_m) \\ & +\omega(f;h_m)\frac{1}{h_m} \parallel G_{m,\mu}((t-x)^2;x) \parallel_\infty^{\frac{1}{2}} \\ & = 2\omega(f;h_m). \end{aligned}$$

Defining the following sets for a given $\varepsilon > 0$:

$$F = \{m \in N : p_m|G_{m,\mu}(f;x) - f(x)| \geq \varepsilon\},$$

$$F_1 = \{m \in N : p_m\omega(f;h_m) \geq \frac{\varepsilon}{2}\},$$

it is easy to see that $F \subset F_1$, and

$$\frac{1}{u_nP_n}\sum_{k=0}^{n}p_k\sum_{m\in F}a_{k,m} \leq \frac{1}{u_nP_n}\sum_{k=0}^{n}p_k\sum_{m\in F_1}a_{k,m}.$$

Hence,

$$S_A^{\bar{N}} - o(u_n) - \lim_n \parallel G_{n,\mu}(f;x) - f(x) \parallel_\infty = 0.$$

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