International Journal of Information and Computation Technology.
ISSN 0974-2239 Volume 4, Number 1 (2014), pp. 27-32

http://www. irphouse.com /ijict.htm

What is Representation Theory?

Gautam Borisagar

Department of Mathematics, Zakir Husain Delhi College, J.L.N. Marg, New Delhi-02.

Abstract

We will define in this article basic concepts in representation theory. Assumption is that reader knows fundamentals of Group Theory and linear algebra. We also relate representation theory with analysis. Some advanced topics are touched in the last section.

1. Introduction

First correct statement about representation theory is that it is that it is initially a game between Group and Vector spaces.In mathematics most of the times we find ourselves working with sets and functions together with more structures or conditions on them. Through our the article G stands for a group. What are possible ways to relate G with a vector space V? Relate corresponds to function. There are not many ways actually if we take structures also into consideration. One way to get a "good" map is homomorphism $\rho : G \to V$. But this is too poor and we cannot do much with this. Now we have two options. Either we produce a group G0 from vector space and relate that group with G some way or we produce vector space W from G and study connections of two vector spaces V and W. Representation theory goes with first option. Now ask yourself, "what are most natural groups associated with a vector space?" Other than V itself under vector addition, important group associated with V is set Aut(V) of all automorphism on V under composition of linear maps. "Natural" map from G to Aut(V) is called representation of G in vector space V!

Definition: 1. A representation of G is a pair (ρ, V) where V is a vector space and ρ is a homo- morphism of G to Aut(V).

Equivalently group G acting on a vector space V by automorphisms is called a representation. Here we note that group G is fixed. We talk of representation of a group. What now? The moment we think of a vector space, we have concept of dimension and underlined field. If the vector space V involved here is finite dimensional we say that the representation is finite dimensional. Degree or dimension of representation is the dimension of the vector space. When it is clear from the context we just say ρ is a representation or just that V is a representation. If the vector space is $n-$dimensional over field of complex numbers C then there is no exaggeration in saying that representation theory is linear algebra! In such case we can use the fact that the automorphisms are $n \times n$ invertible matrices.

1.1. Examples

(i) Trivial representation: Consider any field F as a vector space of dimension one over itself. Trivial homomorphism $\rho : G \to F$ defined by $\rho(x) = 1$ for all $x \in G$ is called trivial representation.

(ii) Consider group S_3 of permutations in three symbols {1, 2, 3}. Take three dimensional vector space V with basis $\{v_1, v_2, v_3\}$. A map defined on basis can be linearly extended to whole vector space. The map $\rho : G \to GL(V)$ is defined by $\rho(\sigma)(v_i) = v_{\sigma(i)}$ gives remaining requirement for representation.

(iii) Permutation representation: Above example can be generalized as follows: Let a group G be given which acts on a set X. Take vector space V which has basis $\{v_x\}_{x \in X}$ parametrized by X. Define $\rho : G \to GL(V)$ using $\rho(g)v_x = v_{g \cdot x}$. This is called permutation representation associated to given action.

(iv) In particular if H is a subgroup of G (not necessarily normal subgroup) then G acts on the set of left cosets $\{gH \mid g \in G\}$. This gives a representation of G.

(v) Character: Since any non-zero complex number corresponds to an automorphism of C, a homomorphism $\chi : G \to C^*$ defines a one dimensional representation. Such representations are also known as characters. For example S_3 has two characters $\chi_0 \equiv 1$ and χ_1 defined by $\chi_1(t)=1$ if t is even permutation and -1 if t is odd permutation.

2. Subrepresentation and Intertwining Maps

The moment we study some new object, we are interested in sub-object also. For example, set-subset, group-subgroup, vector space - vector subspace, ring-subring etc. What is analogous to sub-representation? You might think of a sub group once, but note that we are given a fixed group G and talking about its representations. We are already given a representation, say, (V, ρ) and we want to talk of its sub-representation. So sub-representation must be some(W, ρ_0) satisfying some"natural " condition. Now we define

Definition: 2. Let G be a group. Let (V, ρ) be representation of G. A subrepresentation of V is a subspace W of V such that the restriction $\rho|_W$ defines an automorphism of W for each $g \in G$.

In other words, a subrepresentation is subspace which itself is a representation of the given group. Equivalently the subspace, invariant under G−action, is subrepresentation.

2.1. Examples.

(i) In § 1.1, example 2 take W as span of $\{v1 + v2 + v3\}$ and it is easily checked that W is a subrepresentation. Note that W is trivial representation when seen as representation of S3 .

(ii) Whole space V and zero space $\{0\}$ are always subrepresentations of V. For one imensional representation V these are the only subrepresentations..

Definition: 3. V is said to be an **irreducible representation** if and only if the only subrepresentations of nonzero V are V itself and $\{0\}$ space..

Can we know all the representations of a given group? (We should be clear about word "all".) What are all finite abelian groups? Such questions are known as classification problems. We try to get answer up to some equality. So we now define "equality" of two representations.

Definition: 4. Let (ρ, V) and (ρ^0, W) be two representations of a group G. A linear map $T : V \to W$ satisfying $T(\rho(g)(v) = \rho^0(g)(T(v))$ for all $g \in G$ and $v \in V$ is called G−equivariant or intertwining map. In addition to above conditions if T is one-one and onto then we say the representations are **isomorphic** or **equivalent**.

Theorem 1. Shur's lemma: If V and W are non-isomorphic irreducible representations of G, then the only intertwining map is zero map.

Proof. Let $T : V \to W$ be intertwining map. We will show if T is nonzero then T is isomorphism. In this situation the image $T(V)$ is W because it is a subrepresentation and W is irreducible. So remains to show T is one one. Kernel is also a sub representation of V which is being irreducible must be $\{0\}$ or V. Now $T(V) = 0 \Longleftrightarrow \ker(T) = V$. This forces $\ker(T) = 0$ and consequently the required claim.

Definition: 5. If (ρ_1, V_1) and (ρ_2, V_2) are two representations of G their vector space direct sum $V_1 \oplus V_2$ can be given a natural representation structure defined by $\rho(g)(v_1, v_2) = (\rho_1(g)v_1, \rho_2(g)v_2)$. This is called direct sum of the representations.

Theorem 2. Corresponding to every subrepresentation W of V , there exists a complementary sub space W 0 which is also a subrepresentation. Proof. Ref: [4] Chapter 1. Theorem 3. Any finite dimensional representation is direct sum of irreducible subrepresentations. Proof. With repeated application of theorem 2 apply induction on dimension.

With above theorem the classification of finite dimensional representation of finite group is reduced to know all irreducible representations only. For example S3 has two characters and one 2−dimensional irreducible representation. Any finite dimensional representation is isomorphic to a finite direct sum of these representations. An interested reader is advised to read the book [4].

3. Character Theory

Here we consider vector spaces over complex numbers only. Many times practically it is not easy to find all irreducible representations of a given group. Still we can talk about the dimensions of all its irreducible representation. This part of representation theory is known as character theory. Remember we have already defined one dimensional representation as character, but this is a different concept. This is a good example of how can a word be overused.

Definition: 6. Let $\rho : G \to GL(V)$ define a finite dimensional complex representation. We can think of $GL(V)$ as invertible matrices. The map $\chi : G \to C$ (defined using trace map) by $\chi(g) = Tr(\rho(g))$ is called the character of representation. Recall that trace of a matrix is sum of all diagonal entries of the matrix.

3.1. Facts

(i) The number of irreducible representations of a finite group is the number of conjugacy classes in it.

(ii) Character characterizes representation, i.e., two representations are isomorphic if and only if their characters are same.

(iii) The value $\chi(e)$ gives the dimension of the representation where e is identity element of G.

(iv) A character is a class function in the sense that $\chi(gag^{-1}) = \chi(a)$ for all $g, a \in G$

4. Some Advanced Topics

In case of infinite group and infinite dimensional vector spaces one puts more structures on group and vector spaces to get useful results. For example, locally compact topological group admits Haar measure, which is used to construct a Hilbert space and one studies representation of the group in this space. Representation theory of compact groups over complex vector space is just like the one of finite groups using Haar measure. Completely different flavour is in modular representation theory where the vector space is over filed of finite characteristic. Modular representation theory is used to classify finite simple groups. In this situation theorems 2 and 3 are not true. There are possibility of indecomposable representations which are neither irreducible nor can be written as direct sum of irreducible sub representations! Brauer developed block theory to understand better way modular representations. If the vector space is over field of characteristic p, we call such representation as mod−p representation. For general finite group we don't have classification of indecomposable mod−p representations. In such cases one studies cokernel representation and socal filteration. The only irreducible mod−p representation of a p−group is trivial representation!

Going to infinite groups, the one that are useful to number theory are $GL_n(F)$ where F is local field([5]) of residual characteristic p. In these cases there are 4 types of irreducible modp− representations. In 1994-95 Barthel and Livne ([2],[1]) classified irreducible mod−p representa-tions of $GL_2(F)$ where F is of residual characteristic p into four categories. Out of which one, namely, supersingular representation was left almost undone. Later in 2003, Breuil [3] classified supersingular representations only for $GL_2(Q_p)$. Classification of supersingular representations of $GL_2(F)$ for most other local fields is still open question. It is also worth pointing out that in [6] Vigneras has proved a bijection between 2 dimensional irreducible $F_{p\ (bar)}$-representations of Weil group and the simple supersingular modules of the pro-p-Iwahori-Hecke $F_{p(bar)}$ algebra. For F non-trivial extension of non archimedian local field Q_p bijection between irreducible $F_{p(bar)}$ -representation and simple right pro-p-Iwahori-Hecke modules given by $I(1)$-invariant is an open problem. Here $I(1)$ is pro-p-Iwahori subgroup.

References

[1] L. Barthel and R. Livn´e, Irreducible modular representations of GL2 of a local field, Duke Math. J. 75 (1994), no. 2, 261–292.

[2] Modular representations of GL2 of a local field: the ordinary, unramified case, J. Number Theory 55 (1995), no. 1, 1–27.

[3] Christophe Breuil, Sur quelques repr´esentations modulaires et p-adiques de GL2 (Qp). I, Compositio Math. 138 (2003), no. 2, 165–188.

[4] Jean-Pierre Serre, Linear representations of finite groups, Springer-Verlag, New York, 1977, Translated from the second French edition by Leonard L. Scott, Graduate Texts in Mathematics, Vol. 42.

[5] Local fields, Graduate Texts in Mathematics, vol. 67, Springer-Verlag, New York, 1979, Translated from the French by Marvin Jay Greenberg.

[6] Marie-France Vign´eras, Representations modulo p of the p-adic group GL(2, F), Compos. Math. 140 (2004), no. 2, 333–358. MR 2027193 (2004m:22028)