International Journal of Mathematics Research.
ISSN 0976-5840 Volume 17, Number 2 (2025), pp. 89-105

http://www.irphouse.com

Eccentric Connectivity Index of Some Total Graphs

Shanu Goyal*[1] and Tanya[2]

[1,2]*Department of Mathematics and Statistics, Banasthali Vidyapith, Banasthali-304022, Rajasthan, India.*
[1]*Email: shanugoyal@banasthali.in;* [2]*Email:tanya09mittal@gmail.com*

Abstract

In this paper, we study the eccentric connectivity index of Gallai total graph, anti-Gallai total graph, $\Gamma-$product of two graphs and $\Delta-$product of two graphs for some particular graphs.

Keywords: total graph, Gallai graph, anti-Gallai graph, Gallai total graph, anti-Gallai total graph.

Mathematics Subject Classification: 05C09, 05C10, 05C76

INTRODUCTION

Graphs are fundamental structures in mathematics and computer science, representing networks of interconnected points (vertices) and lines (edges). The vertex set and the edge set are reprresented as $V(G)$ and $E(G)$ respectively. Two vertices are *adjacent* if they are connected by an edge. Two edges are *incident* if they share a common vertex. The twin vertex of u is u' and vice-versa and the set containing all twin vertices of graph G is $V'(G)$.
For a vertex u of $V(G)$, the *degree* $deg_G(u)$ is defined as the number of edges incident to u in G. For any two vertices u and v of graph G, the *distance d(u, v)* is the length of the shortest path connecting both the vertices. For a given vertex u of $V(G)$ its *eccentricity* $\epsilon_G(u)$ is the largest distance between u and any other vertex v of G. Hence, $\epsilon_G(u) = max_{v\in V(G)}d(u, v)$. The *total eccentricity* $\zeta(G)$ of the graph G is the sum of eccentricities of all vertices of a graph G, as mentioned by Doš̃lic and Saheli [1].

*Corresponding Author.

Definition 1. *"The eccentric connectivity index $\xi(G)$ of a graph G is defined as*

$$\xi(G) = \sum_{u \in V(G)} \epsilon_G(u) deg_G(u)."$$

The eccentric connectivity index, introduced by Sharma et al. [2], is a significant topological descriptor particularly in chemical graph theory, where it helps predict molecular behavior and properties, see [3, 4]. The relationships between the eccentric connectivity index and hypergraphs have been explored, enhancing its applicability in complex structures by Yu et al. [5].

Definition 2. *"The $Gallai\ graph\ operator$, Γ, assigns its $Gallai\ graph$ $\Gamma(G)$ to every graph G from $\mathbb{G}$, whose vertex set is the edge set of G and two vertices are adjacent in $\Gamma(G)$ if and only if the corresponding edges are adjacent, but do not lie on a same triangle in G."*

Definition 3. *"The $anti - Gallai\ graph\ operator$, Δ, assigns its anti-Gallai graph $\Delta(G)$ to every graph G from $\mathbb{G}$, whose vertex set is the edge set of G and two vertices are adjacent in $\Delta(G)$ if and only if the corresponding edges are adjacent and sit on a same triangle in G."*

Gallai [6] employed these structures in his investigation of comparability graphs; Sun [7] had already put out the idea. Sun described a lovely class of ideal graphs using the Gallai graphs. Lakshmanan et al. [8] and Le [9] examine a number of Gallai and anti-Gallai graph features. The figure 1 depicts the Gallai graph $\Gamma(G)$ and the anti-Gallai graph $\Delta(G)$ of G.

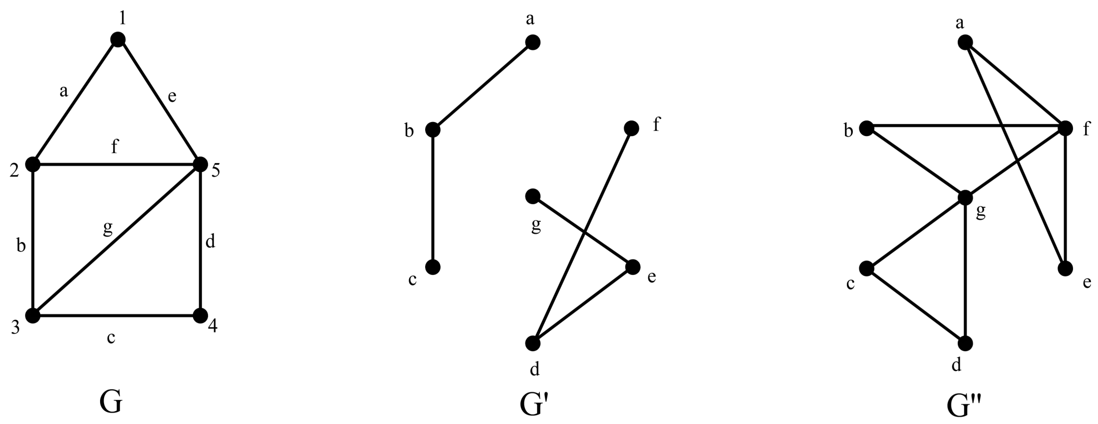


Figure 1: A graph G with its Gallai graph $G' = \Gamma(G)$ and its anti-Gallai graph $G'' = \Delta(G)$

Definition 4. *"The operator L is the $line\ graph\ operator$ assigns its line graph $L(G)$ to every graph G from $\mathbb{G}$, whose vertex set is the edge set of G and two vertices are nearby in $L(G)$ if and only if the corresponding edges are adjacent in G."*

Whitney [10] was the first to study line graphs. Beineke [11, 12] and Chaudhuri [13] have also given consideration to the idea of a line graph.

Definition 5. *"The $total\ graph\ operator$ T assigns the total graph $T(G)$ of every graph G from $\mathbb{G}$, whose vertex set is $V(G) \cup E(G)$ and two vertices are adjacent only if they are adjacent or incident in G."*

Behzad & Chartrand [14] proposed the idea of a complete graph. Characterization of total graphs obtained by Behzad [15]. In the literature, Behzad [16, 17] and Behzad & Radjavi [18, 19] have looked at a number of properties of total graphs. Figure 2 depicts the graph G with its line graph $L(G)$ and its total graph $T(G)$.

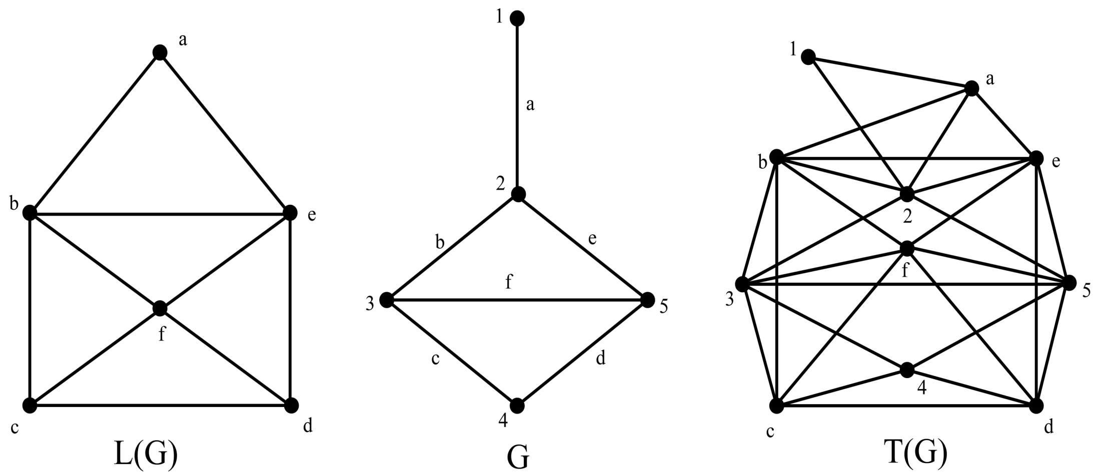


Figure 2: A graph G with its line graph $L(G)$ and its total graph $T(G)$

Definition 6. *"The $Gallai\ total\ graph$ $\Gamma_T(G)$ of a graph $G = (V(G), E(G))$ is a graph with the vertex set $V(\Gamma_T(G)) = V(G) \cup E(G)$ and $uv \in E(\Gamma_T(G))$ if and only if*

- *u and v are adjacent vertices in G, or*
- *u is incident to v or v is incident to u in G, or*
- *u and v are adjacent edges in G which do not span a triangle in G."*

Definition 7. *"The* $anti - Gallai$ $total$ $graph$*,* $\Delta_T(G)$*, of a graph* $G = (V(G), E(G))$ *is a graph with the vertex set* $V(\Delta_T(G)) = V(G) \cup E(G)$ *and* $uv \in E(\Delta_T(G))$ *if and only if*

- u *and* v *are adjacent vertices in G, or*
- u *is incident to* v *or* v *is incident to* u *in G, or*
- u *and* v *are adjacent edges in G and lie on a same triangle in G."*

Goyal et al. [20] defined Gallai total and anti-Gallai total graphs and they also found some results on the traversability of these graphs. Further introduction to Gallai total simplicial complexes was given by Liaquat [21] and characteristics of anti-Gallai graphs were studied by Abbas et al. [22]. Figure 3 demonstrates Gallai total graph $\Gamma_T(G)$ and anti-Gallai total graph $\Delta_T(G)$ of graph G.

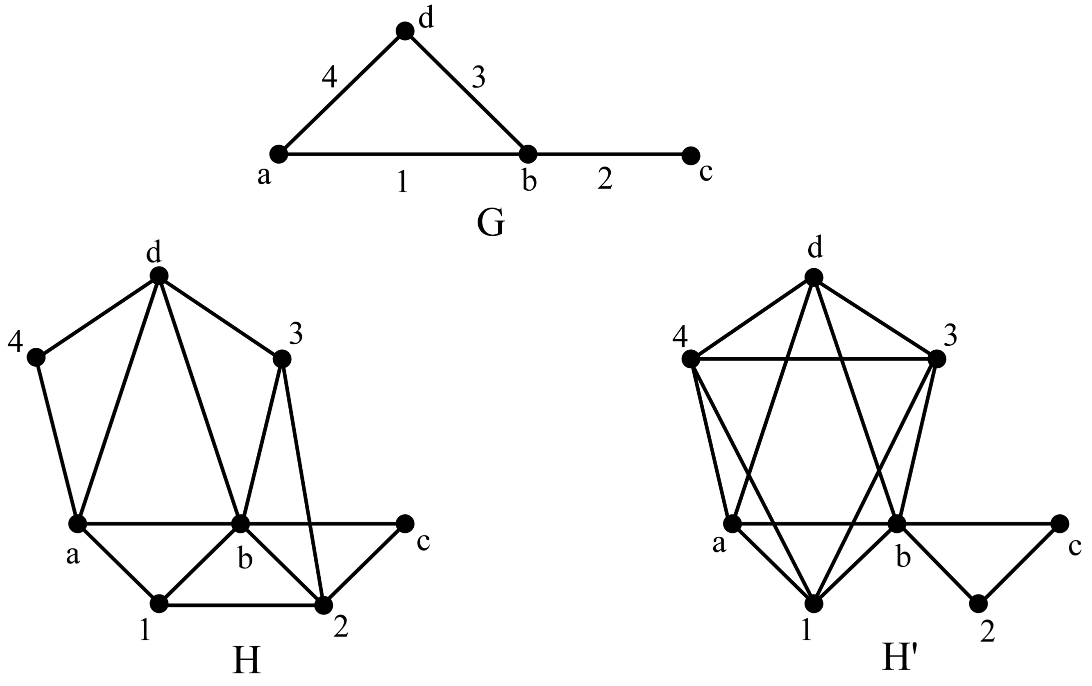


Figure 3: Graph G with its Gallai total graph $H = \Gamma_T(G)$ and anti-Gallai total graph $H' = \Delta_T(G)$

Definition 8. *"Let* $G_1 = (V(G_1), E(G_1))$ *and* $G_2 = (V(G_2), E(G_2))$ *be two non-empty graphs. The* Γ*-product of* G_1 *and* G_2*, denoted by* $\Gamma(G_1)[G_2]$*, is a graph such that* $V(\Gamma(G_1)[G_2]) = (V(G_1) \cup E(G_1)) \times V(G_2)$ *and* $uv \in E(\Gamma(G_1)[G_2])$ *where* $u = (u_1, u_2)$ *and* $v = (v_1, v_2)$ *if and only if*

- $u_1 = v_1 \in E(G_1)$ *and* $u_2v_2 \in E(G_2)$*, or*

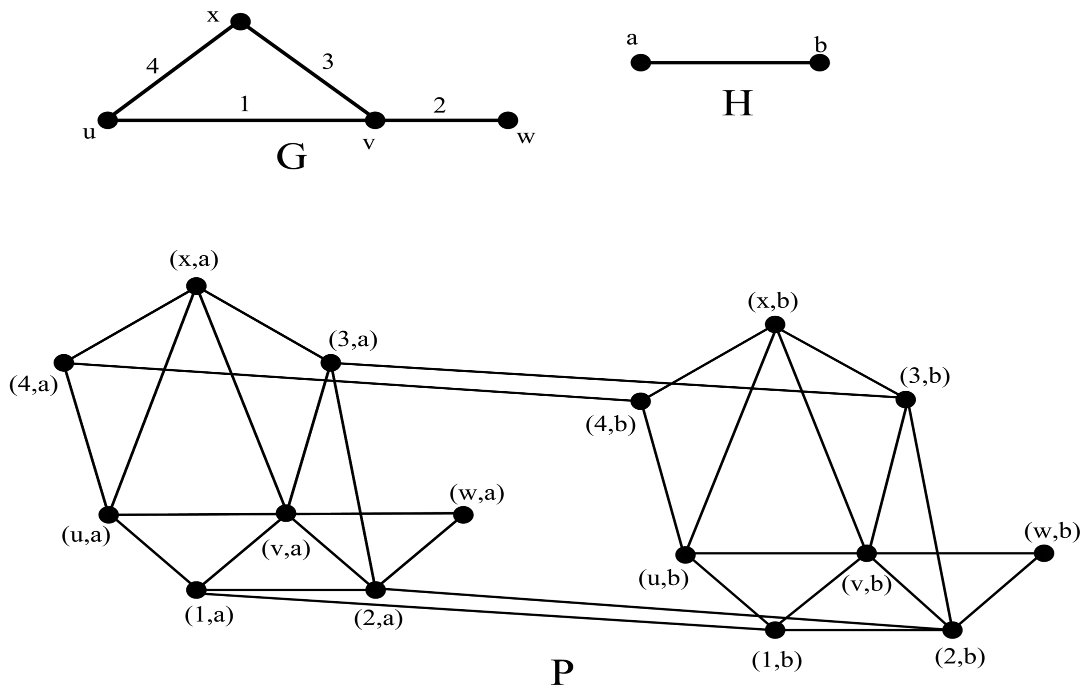
- $u_2 = v_2 \in V(G_2)$ *and* $u_1v_1 \in E(\Gamma_T(G_1))$*."*

Definition 9. *"Let* $G_1 = (V(G_1), E(G_1))$ *and* $G_2 = (V(G_2), E(G_2))$ *be two non-empty graphs. The* Δ*-product of* G_1 *and* G_2*, denoted by* $\Delta(G_1)[G_2]$*, is a graph such that* $V(\Delta(G_1)[G_2]) = (V(G_1) \cup E(G_1)) \times V(G_2)$ *and* $uv \in E(\Delta(G_1)[G_2])$ *where* $u = (u_1, u_2)$ *and* $v = (v_1, v_2)$ *if and only if*

- $u_1 = v_1 \in E(G_1)$ *and* $u_2v_2 \in E(G_2)$*, or*

- $u_2 = v_2 \in V(G_2)$ *and* $u_1v_1 \in E(\Delta_T(G_1))$*."*

Figure 4 demonstrates the Γ-product $\Gamma(G_1)[G_2]$ and Δ-product $\Delta(G_1)[G_2]$ of two graphs G_1 and G_2.



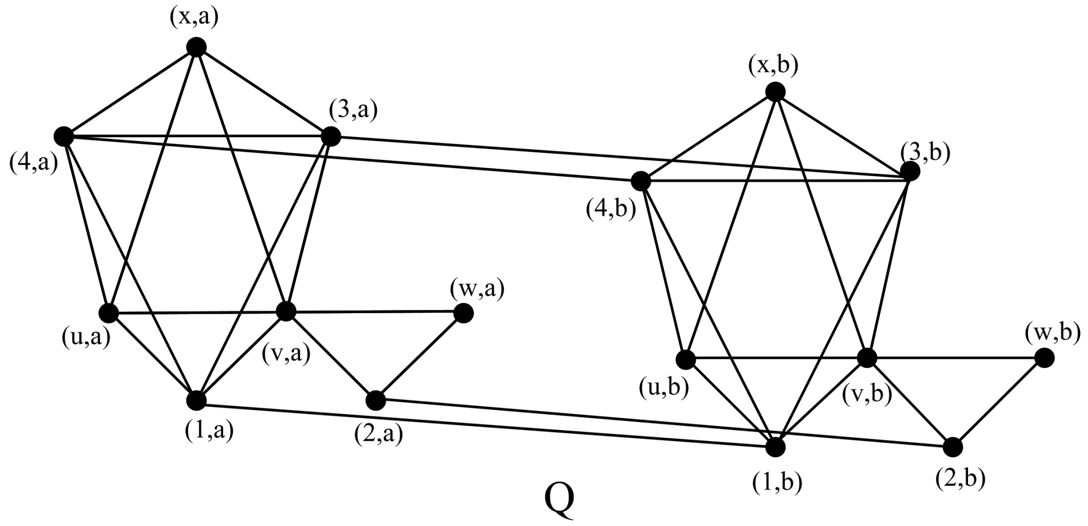


Figure 4: Graphs G and H with its Γ-product $P = \Gamma(G)[H]$ and Δ-product $Q = \Delta(G)[H]$

ECCENTRIC CONNECTIVITY INDEX OF CYCLE GRAPH

Lemma 1. *Let $G = (n, m)$ be a cycle graph with $n \geq 4$. The eccentricity of the vertices of Gallai total graph $\Gamma_T(G)$ is given as*

$$\epsilon_{\Gamma_T(G)}(u) = \begin{cases} \epsilon_G(u) + 1 & ; n \text{ } is \text{ } odd \\ \epsilon_G(u) & ; n \text{ } is \text{ } even. \end{cases}$$

Theorem 1. *The eccentric connectivity index of Gallai total graph $\Gamma_T(G)$ for every cycle graph $G = (n, m)$ and $n \geq 4$ is given by*

$$\xi(\Gamma_T(G)) = \begin{cases} 4\zeta(G) + 4\zeta(L(G)) + 4n + 4m & ; n \text{ } is \text{ } odd \\ 4\zeta(G) + 4\zeta(L(G)) & ; n \text{ } is \text{ } even. \end{cases}$$

Proof. The eccentric connectivity index of $\Gamma_T(G)$,

$$\begin{aligned} \xi(\Gamma_T(G)) &= \sum_{u \in V(\Gamma_T(G))} \epsilon_{\Gamma_T(G)}(u) \; deg_{\Gamma_T(G)}(u) \\ &= \sum_{u \in (V(G) \cup E(G))} \epsilon_{\Gamma_T(G)}(u) \; deg_{\Gamma_T(G)}(u) \end{aligned}$$

For n is odd,

$$\begin{aligned}
\xi(\Gamma_T(G)) &= \sum_{u\in(V(G)\cup E(G))} \epsilon_{\Gamma_T(G)}(u)\ deg_{\Gamma_T(G)}(u) \\
&= \sum_{u\in V(G)} \epsilon_{\Gamma_T(G)}(u)\ deg_{\Gamma_T(G)}(u) + \sum_{u\in E(G)} \epsilon_{\Gamma_T(G)}(u)\ deg_{\Gamma_T(G)}(u) \\
&= \sum_{u\in V(G)} 4\,(\epsilon_G(u)+1) + \sum_{u\in E(G)} 4\,(\epsilon_G(u)+1) \\
&\qquad [By\ Lemma\ 1\ and\ deg_{\Gamma_T(G)}(u)=4] \\
&= 4\sum_{u\in V(G)} (\epsilon_G(u)+1) + 4\sum_{u\in E(G)} (\epsilon_G(u)+1) \\
&= 4\sum_{u\in V(G)} (\epsilon_G(u)+1) + 4\sum_{u\in V(L(G))} (\epsilon_{L(G)}(u)+1) \\
&= 4\zeta(G) + 4\zeta(L(G)) + 4n + 4m
\end{aligned}$$

For n is even,

$$\begin{aligned}
\xi(\Gamma_T(G)) &= \sum_{u\in(V(G)\cup E(G))} \epsilon_{\Gamma_T(G)}(u)\ deg_{\Gamma_T(G)}(u) \\
&= \sum_{u\in V(G)} \epsilon_{\Gamma_T(G)}(u)\ deg_{\Gamma_T(G)}(u) + \sum_{u\in E(G)} \epsilon_{\Gamma_T(G)}(u)\ deg_{\Gamma_T(G)}(u) \\
&= \sum_{u\in V(G)} 4\,(\epsilon_G(u)) + \sum_{u\in E(G)} 4\,(\epsilon_G(u)) \\
&\qquad [By\ Lemma\ 1\ and\ deg_{\Gamma_T(G)}(u)=4] \\
&= 4\sum_{u\in V(G)} (\epsilon_G(u)) + 4\sum_{u\in E(G)} (\epsilon_G(u)) \\
&= 4\sum_{u\in V(G)} \epsilon_G(u) + 4\sum_{u\in V(L(G))} (\epsilon_{L(G)}(u)) \\
&= 4\zeta(G) + 4\zeta(L(G))
\end{aligned}$$

Hence the theorem.

Lemma 2. *Let $G=(n,m)$ be a cycle graph with $n\geq 4$ and odd. The eccentricity of the vertices of anti-Gallai total graph $\Delta_T(G)$ is given as*

$$\epsilon_{\Delta_T(G)}(u) = \epsilon_G(u)+1;\ u\in V(G)\cup E(G).$$

Lemma 3. *Let $G=(n,m)$ be a cycle graph with $n\geq 4$ and even. The eccentricity of the vertices of anti-Gallai total graph $\Delta_T(G)$ is given as*

$$\epsilon_{\Delta_T(G)}(u) = \begin{cases} \epsilon_G(u) & ; u\in V(G) \\ \epsilon_G(u)+1 & ; u\in E(G). \end{cases}$$

Lemma 4. *Let $G = (n, m)$ be a cycle graph with $n \geq 4$. The degree of the vertices of anti-Gallai total graph $\Delta_T(G)$ is given as*

$$deg_{\Delta_T(G)}(u) = \begin{cases} 4 & ; u \in V(G) \\ 2 & ; u \in E(G). \end{cases}$$

Theorem 2. *The eccentric connectivity index of anti-Gallai total graph $\Delta_T(G)$ for every cycle graph $G = (n, m)$ and $n \geq 4$ is given by*

$$\xi(\Delta_T(G)) = \begin{cases} 4\zeta(G) + 2\zeta(L(G)) + 4n + 2m & ; n \ is \ odd \\ 4\zeta(G) + 2\zeta(L(G)) + 2m & ; n \ is \ even. \end{cases}$$

Proof. The eccentric connectivity index of $\Delta_T(G)$,

$$\begin{aligned} \xi(\Delta_T(G)) &= \sum_{u \in V(\Delta_T(G))} \epsilon_{\Delta_T(G)}(u) \ deg_{\Delta_T(G)}(u) \\ &= \sum_{u \in (V(G) \cup E(G))} \epsilon_{\Delta_T(G)}(u) \ deg_{\Delta_T(G)}(u) \end{aligned}$$

For n is odd,

$$\begin{aligned} \xi(\Delta_T(G)) &= \sum_{u \in (V(G) \cup E(G))} \epsilon_{\Delta_T(G)}(u) \ deg_{\Delta_T(G)}(u) \\ &= \sum_{u \in V(G)} \epsilon_{\Delta_T(G)}(u) \ deg_{\Delta_T(G)}(u) + \sum_{u \in E(G)} \epsilon_{\Delta_T(G)}(u) \ deg_{\Delta_T(G)}(u) \\ &= \sum_{u \in V(G)} 4 \ (\epsilon_G(u) + 1) + \sum_{u \in E(G)} 2 \ (\epsilon_G(u) + 1) \\ & \qquad [By \ Lemma \ 2 \ and \ Lemma \ 4] \\ &= 4 \sum_{u \in V(G)} (\epsilon_G(u) + 1) + 2 \sum_{u \in E(G)} (\epsilon_G(u) + 1) \\ &= 4 \sum_{u \in V(G)} (\epsilon_G(u) + 1) + 2 \sum_{u \in V(L(G))} (\epsilon_{L(G)}(u) + 1) \\ &= 4\zeta(G) + 2\zeta(L(G)) + 4n + 2m \end{aligned}$$

For n is even,

$$\begin{aligned}
\xi(\Delta_T(G)) &= \sum_{u\in(V(G)\cup E(G))} \epsilon_{\Delta_T(G)}(u)\ deg_{\Delta_T(G)}(u) \\
&= \sum_{u\in V(G)} \epsilon_{\Delta_T(G)}(u)\ deg_{\Delta_T(G)}(u) + \sum_{u\in E(G)} \epsilon_{\Delta_T(G)}(u)\ deg_{\Delta_T(G)}(u) \\
&= \sum_{u\in V(G)} 4\ (\epsilon_G(u)) + \sum_{u\in E(G)} 2\ (\epsilon_G(u)+1) \\
&\qquad\qquad\qquad\qquad [By\ Lemma\ 3\ and\ Lemma\ 4] \\
&= 4\sum_{u\in V(G)} \epsilon_G(u) + 2\sum_{u\in V(L(G))} (\epsilon_{L(G)}(u)+1) \\
&= 4\zeta(G) + 2\zeta(L(G)) + 2m
\end{aligned}$$

Hence the theorem.

Lemma 5. *Let $G=(n,m)$ be a cycle graph with $n\geq 4$ and odd and $H=(p,q)$ be a connected graph. The eccentricity of the vertices of $\Gamma-$product graph $\Gamma(G)[H]$ is given as*

$$\epsilon_{\Gamma(G)[H]}(u,v) = \epsilon_{\Gamma_T(G)}(u) + \epsilon_H(v);\ u\in(V(G)\times V(H))\cup(E(G)\times V(H)).$$

Lemma 6. *Let $G=(n,m)$ be a cycle graph with $n\geq 4$ and even and $H=(p,q)$ be a connected graph. The eccentricity of the vertices of $\Gamma-$product graph $\Gamma(G)[H]$ is given as*

$$\epsilon_{\Gamma(G)[H]}(u,v) = \begin{cases} \epsilon_{\Gamma_T(G)}(u) + \epsilon_H(v) + 1 & ; u\in V(G)\times V(H) \\ \epsilon_{\Gamma_T(G)}(u) + \epsilon_H(v) & ; u\in E(G)\times V(H). \end{cases}$$

Lemma 7. *Let $G=(n,m)$ be a cycle graph with $n\geq 4$ and $H=(p,q)$ be a connected graph. The degree of the vertices of $\Gamma-$product graph $\Gamma(G)[H]$ is given as*

$$deg_{\Gamma(G)[H]}(u,v) = \begin{cases} 4 & ; u\in V(G) \\ 4 + deg_H(v) & ; u\in E(G). \end{cases}$$

Theorem 3. *The eccentric connectivity index of $\Gamma-$product graph $\Gamma(G)[H]$ for every cycle graph $G=(n,m)$ and $n\geq 4$ and $H=(p,q)$ be a connected graph is given by*

$$\xi(\Gamma(G)[H]) = \begin{cases} 4p\zeta(\Gamma_T(G)) + 4(n+m)\zeta(H) + 2q[\zeta(\Gamma_T(G)) - \zeta(G) - n] \\ +m\xi(H); n\ is\ odd \\ 4p\zeta(\Gamma_T(G)) + 4(n+m)\zeta(H) + 4np + 2q[\zeta(\Gamma_T(G)) - \zeta(G)] \\ +m\xi(H); n\ is\ even. \end{cases}$$

Proof. The eccentric connectivity index of $\Gamma(G)[H]$,

$$\begin{aligned}\xi(\Gamma(G)[H]) &= \sum_{(u,v)\in V(\Gamma(G)[H])} \epsilon_{\Gamma(G)[H]}(u,v)\ deg_{\Gamma(G)[H]}(u,v)\\ &= \sum_{(u,v)\in((V(G)\cup E(G))\times V(H))} \epsilon_{\Gamma(G)[H]}(u,v)\ deg_{\Gamma(G)[H]}(u,v)\end{aligned}$$

For n *is odd,*

$$\begin{aligned}\xi(\Gamma(G)[H]) &= \sum_{(u,v)\in((V(G)\cup E(G))\times V(H))} \epsilon_{\Gamma(G)[H]}(u,v)\ deg_{\Gamma(G)[H]}(u,v)\\ &= \sum_{(u,v)\in(V(G)\times V(H))} \epsilon_{\Gamma(G)[H]}(u,v)\ deg_{\Gamma(G)[H]}(u,v)\\ &+ \sum_{(u,v)\in(E(G)\times V(H))} \epsilon_{\Gamma(G)[H]}(u,v)\ deg_{\Gamma(G)[H]}(u,v)\\ &= \sum_{(u,v)\in(V(G)\times V(H))} 4(\epsilon_{\Gamma_T(G)}(u)+\epsilon_H(v))\\ &+ \sum_{(u,v)\in(E(G)\times V(H))} (4+deg_H(v))(\epsilon_{\Gamma_T(G)}(u)+\epsilon_H(v))\end{aligned}$$

$$[By\ Lemma\ 5\ and\ Lemma\ 7]$$

$$= 4\sum_{u\in V(G)}\epsilon_{\Gamma_T(G)}(u)\sum_{v\in V(H)}1 + 4\sum_{u\in V(G)}1\sum_{v\in V(H)}\epsilon_H(v) + 4\sum_{u\in E(G)}\epsilon_{\Gamma_T(G)}(u)\sum_{v\in V(H)}1$$
$$+ 4\sum_{u\in E(G)}1\sum_{v\in V(H)}\epsilon_H(v) + \sum_{u\in E(G)}\epsilon_{\Gamma_T(G)}(u)\sum_{v\in V(H)}deg_H(v) + \sum_{u\in E(G)}1\sum_{v\in V(H)}deg_H(v)\epsilon_H(v)$$
$$= 4p\zeta(\Gamma_T(G)) + 4(n+m)\zeta(H) + 2q[\zeta(\Gamma_T(G)) - \zeta(G) - n] + m\xi(H)$$

For n *is even,*

$$\begin{aligned}\xi(\Gamma(G)[H]) &= \sum_{(u,v)\in((V(G)\cup E(G))\times V(H))} \epsilon_{\Gamma(G)[H]}(u,v)\ deg_{\Gamma(G)[H]}(u,v)\\ &= \sum_{(u,v)\in(V(G)\times V(H))} \epsilon_{\Gamma(G)[H]}(u,v)\ deg_{\Gamma(G)[H]}(u,v)\\ &+ \sum_{(u,v)\in(E(G)\times V(H))} \epsilon_{\Gamma(G)[H]}(u,v)\ deg_{\Gamma(G)[H]}(u,v)\\ &= \sum_{(u,v)\in(V(G)\times V(H))} 4(\epsilon_{\Gamma_T(G)}(u)+\epsilon_H(v)+1)\\ &+ \sum_{(u,v)\in(E(G)\times V(H))} (4+deg_H(v))(\epsilon_{\Gamma_T(G)}(u)+\epsilon_H(v))\end{aligned}$$

$$[By\ Lemma\ 6\ and\ Lemma\ 7]$$

$$= 4\sum_{u\in V(G)} \epsilon_{\Gamma_T(G)}(u) \sum_{v\in V(H)} 1 + 4\sum_{u\in V(G)} 1 \sum_{v\in V(H)} \epsilon_H(v) + \sum_{u\in V(G)} 1 \sum_{v\in V(H)} 1$$
$$+ 4\sum_{u\in E(G)} \epsilon_{\Gamma_T(G)}(u) \sum_{v\in V(H)} 1 + 4\sum_{u\in E(G)} 1 \sum_{v\in V(H)} \epsilon_H(v)$$
$$+ \sum_{u\in E(G)} \epsilon_{\Gamma_T(G)}(u) \sum_{v\in V(H)} deg_H(v) + \sum_{u\in E(G)} 1 \sum_{v\in V(H)} deg_H(v)\epsilon_H(v)$$
$$= 4p\zeta(\Gamma_T(G)) + 4(n+m)\zeta(H) + 4np + 2q[\zeta(\Gamma_T(G)) - \zeta(G)] + m\xi(H)$$

Hence the theorem.

Lemma 8. *Let $G = (n, m)$ be a cycle graph with $n \geq 4$ and odd and $H = (p, q)$ be a connected graph. The eccentricity of the vertices of $\Delta-$product graph $\Delta(G)[H]$ is given as*

$$\epsilon_{\Delta(G)[H]}(u, v) = \epsilon_{\Delta_T(G)}(u) + \epsilon_H(v);\ u \in (V(G)\times V(H)) \cup (E(G)\times V(H)).$$

Lemma 9. *Let $G = (n, m)$ be a cycle graph with $n \geq 4$ and even and $H = (p, q)$ be a connected graph. The eccentricity of the vertices of $\Delta-$product graph $\Delta(G)[H]$ is given as*

$$\epsilon_{\Delta(G)[H]}(u, v) = \begin{cases} \epsilon_{\Delta_T(G)}(u) + \epsilon_H(v) + 1 & ; u \in V(G)\times V(H) \\ \epsilon_{\Delta_T(G)}(u) + \epsilon_H(v) & ; u \in E(G)\times V(H). \end{cases}$$

Lemma 10. *Let $G = (n, m)$ be a cycle graph with $n \geq 4$ and $H = (p, q)$ be a connected graph. The degree of the vertices of $\Delta-$product graph $\Delta(G)[H]$ is given as*

$$deg_{\Delta(G)[H]}(u, v) = \begin{cases} 4 & ; u \in V(G) \\ 2 + deg_H(v) & ; u \in E(G). \end{cases}$$

Theorem 4. *The eccentric connectivity index of $\Delta-$product graph $\Delta(G)[H]$ for every cycle graph $G = (n, m)$ and $n \geq 4$ and $H = (p, q)$ be a connected graph is given by*

$$\xi(\Delta(G)[H]) = \begin{cases} 2(2p+q)\zeta(\Delta_T(G)) - 2(p+q)\zeta(G) - 2n(p+q) + 2(2n+m)\zeta(H) \\ +m\xi(H); n\ is\ odd \\ 2(p+q)\zeta(\Delta_T(G)) + 2(p-q)\zeta(G) + 4np + 2(2n+m)\zeta(G) \\ +m\xi(H); n\ is\ even. \end{cases}$$

Proof. The eccentric connectivity index of $\Delta(G)[H]$,

$$\xi(\Delta(G)[H]) = \sum_{(u,v)\in V(\Delta(G)[H])} \epsilon_{\Delta(G)[H]}(u, v)\ deg_{\Delta(G)[H]}(u, v)$$
$$= \sum_{(u,v)\in((V(G)\cup E(G))\times V(H))} \epsilon_{\Delta(G)[H]}(u, v)\ deg_{\Delta(G)[H]}(u, v)$$

For n is odd,

$$\begin{aligned}\xi(\Delta(G)[H]) &= \sum_{(u,v)\in((V(G)\cup E(G))\times V(H))} \epsilon_{\Delta(G)[H]}(u,v)\ deg_{\Delta(G)[H]}(u,v)\\ &= \sum_{(u,v)\in(V(G)\times V(H))} \epsilon_{\Delta(G)[H]}(u,v)\ deg_{\Delta(G)[H]}(u,v)\\ &+ \sum_{(u,v)\in(E(G)\times V(H))} \epsilon_{\Delta(G)[H]}(u,v)\ deg_{\Delta(G)[H]}(u,v)\\ &= \sum_{(u,v)\in(V(G)\times V(H))} 4(\epsilon_{\Gamma_T(G)}(u)+\epsilon_H(v))\\ &+ \sum_{(u,v)\in(E(G)\times V(H))} (2+deg_H(v))(\epsilon_{\Delta_T(G)}(u)+\epsilon_H(v))\end{aligned}$$

$$[By\ Lemma\ 8\ and\ Lemma\ 10]$$

$$= 4\sum_{u\in V(G)}\epsilon_{\Delta_T(G)}(u)\sum_{v\in V(H)}1 + 4\sum_{u\in V(G)}1\sum_{v\in V(H)}\epsilon_H(v) + 2\sum_{u\in E(G)}\epsilon_{\Delta_T(G)}(u)\sum_{v\in V(H)}1$$

$$+ 2\sum_{u\in E(G)}1\sum_{v\in V(H)}\epsilon_H(v) + \sum_{u\in E(G)}\epsilon_{\Delta_T(G)}(u)\sum_{v\in V(H)}deg_H(v) + \sum_{u\in E(G)}1\sum_{v\in V(H)}deg_H(v)\epsilon_H(v)$$

$$= 2(2p+q)\zeta(\Delta_T(G)) - 2(p+q)\zeta(G) - 2n(p+q) + 2(2n+m)\zeta(H) + m\xi(H)$$

For n is even,

$$\begin{aligned}\xi(\Delta(G)[H]) &= \sum_{(u,v)\in((V(G)\cup E(G))\times V(H))} \epsilon_{\Delta(G)[H]}(u,v)\ deg_{\Delta(G)[H]}(u,v)\\ &= \sum_{(u,v)\in(V(G)\times V(H))} \epsilon_{\Delta(G)[H]}(u,v)\ deg_{\Delta(G)[H]}(u,v)\\ &+ \sum_{(u,v)\in(E(G)\times V(H))} \epsilon_{\Delta(G)[H]}(u,v)\ deg_{\Delta(G)[H]}(u,v)\\ &= \sum_{(u,v)\in(V(G)\times V(H))} 4(\epsilon_{\Delta_T(G)}(u)+\epsilon_H(v)+1)\\ &+ \sum_{(u,v)\in(E(G)\times V(H))} (2+deg_H(v))(\epsilon_{\Delta_T(G)}(u)+\epsilon_H(v))\end{aligned}$$

$$[By\ Lemma\ 9\ and\ Lemma\ 10]$$

$$= 4\sum_{u\in V(G)}\epsilon_{\Delta_T(G)}(u)\sum_{v\in V(H)}1 + 4\sum_{u\in V(G)}1\sum_{v\in V(H)}\epsilon_H(v) + \sum_{u\in V(G)}1\sum_{v\in V(H)}1$$

$$+ 2\sum_{u\in E(G)}\epsilon_{\Delta_T(G)}(u)\sum_{v\in V(H)}1 + 2\sum_{u\in E(G)}1\sum_{v\in V(H)}\epsilon_H(v)$$

$$+ \sum_{u\in E(G)}\epsilon_{\Delta_T(G)}(u)\sum_{v\in V(H)}deg_H(v) + \sum_{u\in E(G)}1\sum_{v\in V(H)}deg_H(v)\epsilon_H(v)$$

$$= 2(p+q)\zeta(\Delta_T(G)) + 2(p-q)\zeta(G) + 4np + 2(2n+m)\zeta(G) + m\xi(H)$$

Hence the theorem.

ECCENTRIC CONNECTIVITY INDEX OF REGULAR GRAPH

Theorem 5. *The eccentric connectivity index of Gallai total graph $\Gamma_T(G)$ for every regular graph $G = (n, m)$ and $n \geq 4$ is given by*

$$\xi(\Gamma_T(G)) = 4(n-1)\zeta(G) + 3n(n-1).$$

Proof. The eccentric connectivity index of $\Gamma_T(G)$,

$$\begin{aligned}
\xi(\Gamma_T(G)) &= \sum_{u \in V(\Gamma_T(G))} \epsilon_{\Gamma_T(G)}(u)\ deg_{\Gamma_T(G)}(u) \\
&= \sum_{u \in (V(G) \cup E(G))} \epsilon_{\Gamma_T(G)}(u)\ deg_{\Gamma_T(G)}(u) \\
&= \sum_{u \in V(G)} \epsilon_{\Gamma_T(G)}(u)\ deg_{\Gamma_T(G)}(u) + \sum_{u \in E(G)} \epsilon_{\Gamma_T(G)}(u)\ deg_{\Gamma_T(G)}(u) \\
&= 4(n-1) \sum_{u \in V(G)} \epsilon_G(u) + 3 \sum_{u \in E(G)} deg_{\Gamma_T(G)}(u) \\
&= 4(n-1)\zeta(G) + 3\left(\frac{2n(n-1)}{2}\right) \\
&= 4(n-1)\zeta(G) + 3n(n-1)
\end{aligned}$$

Hence the theorem.

Theorem 6. *The eccentric connectivity index of anti-Gallai total graph $\Delta_T(G)$ for every regular graph $G = (n, m)$ and $n \geq 4$ is given by*

$$\xi(\Delta_T(G)) = 4(n-1)\zeta(G) + 2n(n-1)^2.$$

Proof. The eccentric connectivity index of $\Delta_T(G)$,

$$\begin{aligned}
\xi(\Delta_T(G)) &= \sum_{u \in V(\Delta_T(G))} \epsilon_{\Delta_T(G)}(u)\ deg_{\Delta_T(G)}(u) \\
&= \sum_{u \in (V(G) \cup E(G))} \epsilon_{\Delta_T(G)}(u)\ deg_{\Delta_T(G)}(u) \\
&= \sum_{u \in V(G)} \epsilon_{\Delta_T(G)}(u)\ deg_{\Delta_T(G)}(u) + \sum_{u \in E(G)} \epsilon_{\Delta_T(G)}(u)\ deg_{\Delta_T(G)}(u) \\
&= 2(n-1) \sum_{u \in V(G)} \epsilon_G(u) + 2 \sum_{u \in E(G)} deg_{\Delta_T(G)}(u) \\
&= 2(n-1)(2\zeta(G)) + 2\left(\frac{2n(n-1)^2}{2}\right) \\
&= 4(n-1)\zeta(G) + 2n(n-1)^2
\end{aligned}$$

Hence the theorem.

Theorem 7. *The eccentric connectivity index of* $\Gamma-$*product graph* $\Gamma(G)[H]$ *for every regular graph* $G=(n,m)$ *and* $n\geq 4$ *and* $H=(p,q)$ *be a connected graph is given by*

$$\xi(\Gamma(G)[H])=2(p+q)\zeta(\Gamma_T(G))+4(np-2p+q)\zeta(G)+2(n^2-n+m)\zeta(H)+m\xi(H).$$

Proof. The eccentric connectivity index of $\Gamma(G)[H]$,

$$\begin{aligned}
\xi(\Gamma(G)[H]) &= \sum_{(u,v)\in V(\Gamma(G)[H])} \epsilon_{\Gamma(G)[H]}(u,v)\ deg_{\Gamma(G)[H]}(u,v)\\
&= \sum_{(u,v)\in((V(G)\cup E(G))\times V(H))} \epsilon_{\Gamma(G)[H]}(u,v)\ deg_{\Gamma(G)[H]}(u,v)\\
&= \sum_{(u,v)\in(V(G)\times V(H))} \epsilon_{\Gamma(G)[H]}(u,v)\ deg_{\Gamma(G)[H]}(u,v)\\
&+ \sum_{(u,v)\in(E(G)\times V(H))} \epsilon_{\Gamma(G)[H]}(u,v)\ deg_{\Gamma(G)[H]}(u,v)\\
&= \sum_{(u,v)\in(V(G)\times V(H))} 2(n-1)\left(\epsilon_{\Gamma_T(G)}(u)+\epsilon_H(v)\right)\\
&+ \sum_{(u,v)\in(E(G)\times V(H))} (2+deg_H(v))\left(\epsilon_{\Gamma_T(G)}(u)+\epsilon_H(v)\right)\\
&= 2(n-1)\sum_{u\in V(G)}\epsilon_{\Gamma_T(G)}(u)\sum_{v\in V(H)}1+2(n-1)\sum_{u\in V(G)}1\sum_{v\in V(H)}\epsilon_H(v)\\
&+2\sum_{u\in E(G)}\epsilon_{\Gamma_T(G)}(u)\sum_{v\in V(H)}1+2\sum_{u\in E(G)}1\sum_{v\in V(H)}\epsilon_H(v)\\
&+\sum_{u\in E(G)}\epsilon_{\Gamma_T(G)}(u)\sum_{v\in V(H)}deg_H(v)+\sum_{u\in E(G)}1\sum_{v\in V(H)}deg_H(v)\epsilon_H(v)\\
&= 2(p+q)\zeta(\Gamma_T(G))+4(np-2p+q)\zeta(G)+2(n^2-n+m)\zeta(H)+m\xi(H)
\end{aligned}$$

Hence the theorem.

Theorem 8. *The eccentric connectivity index of* $\Delta-$*product graph* $\Delta(G)[H]$ *for every regular graph* $G=(n,m)$ *and* $n\geq 4$ *and* $H=(p,q)$ *be a connected graph is given by*

$$\begin{aligned}
\xi(\Delta(G)[H]) = 2q\zeta(\Delta_T(G))+4(np-p-q)\zeta(G)+(n^3-n)\zeta(H)\\
+p\left(\xi(\Delta_T(G)-4\xi(G)\right)+m\xi(H).
\end{aligned}$$

Proof. The eccentric connectivity index of $\Delta(G)[H]$,

$$\begin{aligned}
\xi(\Delta(G)[H]) &= \sum_{(u,v)\in V(\Delta(G)[H])} \epsilon_{\Delta(G)[H]}(u,v)\ deg_{\Delta(G)[H]}(u,v) \\
&= \sum_{(u,v)\in((V(G)\cup E(G))\times V(H))} \epsilon_{\Delta(G)[H]}(u,v)\ deg_{\Delta(G)[H]}(u,v) \\
&= \sum_{(u,v)\in(V(G)\times V(H))} \epsilon_{\Delta(G)[H]}(u,v)\ deg_{\Delta(G)[H]}(u,v) \\
&+ \sum_{(u,v)\in(E(G)\times V(H))} \epsilon_{\Delta(G)[H]}(u,v)\ deg_{\Delta(G)[H]}(u,v) \\
&= \sum_{(u,v)\in(V(G)\times V(H))} 2(n-1)\left(\epsilon_{\Delta_T(G)}(u)+\epsilon_H(v)\right) \\
&+ \sum_{(u,v)\in(E(G)\times V(H))} \left(deg_{\Delta_T(G)}+deg_H(v)\right)\left(\epsilon_{\Delta_T(G)}(u)+\epsilon_H(v)\right) \\
&= 2(n-1)\sum_{u\in V(G)} \epsilon_{\Delta_T(G)}(u)\sum_{v\in V(H)} 1 + 2(n-1)\sum_{u\in V(G)} 1 \sum_{v\in V(H)} \epsilon_H(v) \\
&+ \sum_{u\in E(G)} \epsilon_{\Delta_T(G)}(u) deg_{\Delta_T(G)}(u)\sum_{v\in V(H)} 1 + \sum_{u\in E(G)} \epsilon_{\Delta_T(G)}(u)\sum_{v\in V(H)} deg_H(v) \\
&+ \sum_{u\in E(G)} deg_{\Delta_T(G)}(u)\sum_{v\in V(H)} \epsilon_H(v) + \sum_{u\in E(G)} 1 \sum_{v\in V(H)} deg_H(v)\epsilon_H(v) \\
&= 2q\zeta(\Delta_T(G)) + 4(np-p-q)\zeta(G) + (n^3-n)\zeta(H) + m\xi(H) \\
&+ p\left(\xi(\Delta_T(G)) - 4\xi(G)\right)
\end{aligned}$$

Hence the theorem.

CONCLUSION

In the paper, we have derived the Eccentric Connectivity Index of Gallai total graph and anti-Gallai graph, $\Gamma-$product graph, and $\Delta-$product graph for some particular graphs, namely cycle and regular graphs.

REFERENCES

[1] Tomislav Došlic and Mahboubeh Saheli. Eccentric connectivity index of composite graphs. *Util. Math*, 95:3–22, 2014.

[2] Vikas Sharma, Reena Goswami, and AK Madan. Eccentric connectivity index: A novel highly discriminating topological descriptor for structure- property

and structure- activity studies. *Journal of chemical information and computer sciences*, 37(2):273–282, 1997.

[3] A Berin Greeni and S Jancy. Eccentric connectivity index of nanosheets and nanotube of sio2. *Journal of Advanced Computational Intelligence and Intelligent Informatics*, 27(4):649–654, 2023.

[4] Rukhsar Zireen, Saima Noureen, and Amir Hussain. Eccentricity based entropy indices of water soluble dendritic unimolecular polyether micelle. *Mathematical Sciences and Applications*, 2(1):14–20, 2023.

[5] Guihai Yu, Renjie Wu, and Xingfu Li. The connective eccentricity index of hypergraphs. *Mathematics*, 10(23):4574, 2022.

[6] Tibor Gallai. Transitiv orientierbare graphen. *Acta Mathematica Hungarica*, 18(1-2):25–66, 1967.

[7] Liping Sun. Two classes of perfect graphs. *Journal of Combinatorial Theory, Series B*, 53(2):273–292, 1991.

[8] Aparna Lakshmanan S, SB Rao, and A Vijayakumar. Gallai and anti-gallai graphs of a graph. *Mathematica Bohemica*, 132(1):43–54, 2007.

[9] Van Bang Le. Gallai graphs and anti-gallai graphs. *Discrete Mathematics*, 159(1-3):179–189, 1996.

[10] Hassler Whitney. Congruent graphs and the connectivity of graphs. *Hassler Whitney Collected Papers*, pages 61–79, 1992.

[11] Lowell W Beineke. Derived graphs and digraphs. *Beiträge zur graphentheorie*, pages 17–33, 1968.

[12] Lowell W Beineke. Characterizations of derived graphs. *Journal of Combinatorial theory*, 9(2):129–135, 1970.

[13] DK Ray-Chaudhuri. Characterization of line graphs. *Journal of Combinatorial Theory*, 3(3):201–214, 1967.

[14] Mehdi Behzad and Gary Chartrand. Total graphs and traversability. *Proceedings of the Edinburgh Mathematical Society*, 15(2):117–120, 1966.

[15] M Behzad. A characterization of total graphs. *Proceedings of the American Mathematical Society*, 26(3):383–389, 1970.

[16] Mehdi Behzad. A criterion for the planarity of the total graph of a graph. *Mathematical Proceedings of the Cambridge Philosophical Society*, 63(3):679–681, 1967.

[17] Mehdi Behzad. The connectivity of total graphs. *Bulletin of the Australian Mathematical Society*, 1(2):175–181, 1969.

[18] Mehdi Behzad and Heydar Radjavi. The total group of a graph. *Proceedings of the American Mathematical Society*, 19(1):158–163, 1968.

[19] Mehdi Behzad and Heydar Radjavi. Structure of regular total graphs. *Journal of the London Mathematical Society*, 1(1):433–436, 1969.

[20] Pravin Garg, Deepa Sinha, and Shanu Goyal. Eulerian and hamiltonian properties of gallai and anti-gallai total graphs. *Journal of the Indonesian Mathematical Society*, 21(2):105–116, 2015.

[21] Adnan Liaquat. Characterizations of gallai total simplicial complexes. 2018.

[22] Najam Ul Abbas, Imran Ahmed, and Ayesha Kiran. Cohen-macaulayness of total simplicial complexes. *arXiv preprint arXiv:2310.09966*, 2023.