International Journal of Mathematics Research.
ISSN 0976-5840 Volume 17, Number 2 (2025), pp. 107-114

http://www.irphouse.com

Jordan k-Derivations on Gamma Rings

[1*]**Rajib Das,**
[1]Department of Mathematics,
Suren Das College (Autonomous), Hajo,
Assam 781102, India
*Corresponding author: drajib1974@gmail.com

[2]**Basistha Ram Bhuyan**
[2]Department of Mathematics,
LCB College, Guwahati 781011,
Assam, India
Email: basistharambhuyan@gmail.com

[3]**Nilima Dutta**
[3]Department of Statistics, Pub-Kamrup College,
Assam 781381, India
Email: nilima_kamrup@gmail.com

Abstract

In this paper, we discuss 'Derivation', 'Generalised Derivation', 'Jordan Derivation', 'k-Derivation', and 'Jordan k-Derivation' in the field of $\Gamma-$rings. We construct many interesting examples and prove some enlightening results.

Key Words: Γ-ring, Γ-ideal, Γ-Derivation, Γ-Jordan k-Derivation 2020 AMS Subject Classification: 13C05, 17D20

1.Introduction: We first highlight a few concepts and fundamental examples in the field of Gamma-rings, [1,2,5,10].

1.1Definition: Let X and Γ be two additive Abelian groups. X is called a Γ- ring if a ternary composition $X \times \Gamma \times X \rightarrow X$ is defined on X, $(x, \alpha, y) \rightarrow x\alpha y$ such that the following axioms are satisfied:

(i) $(x + y)\alpha z = x\alpha z + y\alpha z$, (ii) $x(\alpha + \beta)y = x\alpha y + x\beta y$, (iii) $(x\alpha y)\beta z = x\alpha(y\beta z)$ for all x, y, z $\in$ X and $\alpha, \beta \in \Gamma$
From definition, it follows that $0\alpha y = x\alpha 0 = x0y = 0$

1.2Definition: An additive subgroup I of the $\Gamma-$ring X is called a right (left) ideal of X, if $x\alpha y \in I$ (respectively $y\alpha x \in I$) for all $x \in I$, $\alpha \in \Gamma$, y∈X.

In other words, an additive subgroup I of the $\Gamma-$ring X is called a right (left) ideal of X , if $I\Gamma X \subseteq I$ (respectively $X\Gamma I \subseteq I$)

A left and right $\Gamma-$ideal of X is called a $\Gamma-$ideal of X.

1.3 Definition: A $\Gamma-$ring X is called prime if $x\Gamma X\Gamma y = 0 \Rightarrow x = 0$ or $y = 0 \quad \forall x, y \in X$

1.4Definition: A $\Gamma-$ring X is called semi prime if $x\Gamma X\Gamma x = 0 \Rightarrow x = 0, \quad \forall x \in X$

Remark: Every prime Γ −ring is obviously semi prime.

1.5Definition: Let X be a Γ −ring. An additive mapping $\emptyset : X \to X$ is called a derivation on X if $\emptyset(x\alpha y) = \emptyset(x)\alpha y + x\alpha\emptyset(y) \quad \forall\, x, y \in X, \forall \alpha \in \Gamma$

The zero mapping, $\emptyset(x) = 0, \ \forall\, x \in X$ is always a derivation on any Γ −ring.

Example 1: Let X be a Γ −ring, and the additive mapping $\emptyset : X \to X$ be defined as $\emptyset(x) = x$

Then $\forall\, x, y \in X, \forall \alpha \in \Gamma$

$\emptyset(x\alpha y) = x\alpha y$ and $\emptyset(x)\alpha y + x\alpha\emptyset(y) = x\alpha y + x\alpha y = 2x\alpha y$

since, $\emptyset(x\alpha y) \neq \emptyset(x)\alpha y + x\alpha\emptyset(y)$. Therefore, $\emptyset$ is not a derivation on X.

Example 2: Let X be the collection of all 1×2 rectangular matrices and Γ be the collection of all 2×1 real matrices. Then X is a Γ −ring.

For $x = (x_1 \ x_2),\ y = (y_1 \ y_2) \in X$ and $\alpha = \begin{pmatrix} a \\ b \end{pmatrix} \in \Gamma$

Suppose the additive mapping $\emptyset : X \to X$ be defined as $\emptyset(x) = (ax_1 \ bx_2)$

We have, $x\alpha y = (x_1 \ x_2)\begin{pmatrix} a \\ b \end{pmatrix}(y_1 \ y_2)$

$= (ax_1 + bx_2)(y_1 \ y_2)$

$= ((ax_1 + bx_2)y_1 \ (ax_1 + bx_2)y_2)$

And $\emptyset(x\alpha y) = (a(ax_1 + bx_2)y_1 \ b(ax_1 + bx_2)y_2)$

$= (a^2x_1y_1 + abx_2y_1 \quad abx_1y_2 + b^2x_2\, y_2)$

Again, $\emptyset(x)\alpha y + x\alpha\emptyset(y) = (ax_1 \ bx_2)\begin{pmatrix} a \\ b \end{pmatrix}(y_1 \ y_2) + (x_1 \ x_2)\begin{pmatrix} a \\ b \end{pmatrix}(ay_1 \ by_2)$

$= (a^2x_1 + b^2x_2)(y_1 \ y_2) + (ax_1 + bx_2)(ay_1 \ by_2)$

$= (a^2x_1y_1 + b^2x_2y_1 \quad a^2x_1y_2 + b^2x_2\, y_2) + (a^2x_1y_1 + abx_2y_1 \quad abx_1y_2 + b^2x_2\, y_2)$

$= (2a^2x_1y_1 + b^2x_2y_1 + abx_2y_1 \qquad a^2x_1y_2 + abx_1y_2 + 2b^2x_2\, y_2)$

Now if $\emptyset$ is a derivation on X, then we must have,

$\emptyset(x\alpha y) = \emptyset(x)\alpha y + x\alpha\emptyset(y)$

that is, we must have

$a^2x_1y_1 + abx_2y_1 = 2a^2x_1y_1 + b^2x_2y_1 + abx_2y_1$

that is, $a^2x_1y_1 + b^2x_2y_1 = 0$ ……………… (i)

and $abx_1y_2 + b^2x_2\, y_2 = a^2x_1y_2 + abx_1y_2 + 2b^2x_2\, y_2$

that is, $a^2x_1y_2 + b^2x_2\, y_2 = 0$ ……………… (ii)

(i) and (ii) are the conditions for $\emptyset$ to be a derivation on X.

Example 3: Let X be a Γ −ring and $\emptyset : X \to X$ be a derivation. Consider $X_1 = \{(x, x) \mid x \in X\}$ and $\Gamma_1 = \{(\alpha, \alpha) \mid \alpha \in \Gamma\}$. Define addition and multiplication on X_1 and Γ_1 by $(x_1, x_1) + (x_2, x_2) = (x_1 + x_2,\ x_1 + x_2),\ (x_1, x_1)(\alpha, \alpha)(x_2, x_2) = (x_1\alpha x_2, x_1\alpha x_2)$ and $(\alpha_1, \alpha_1) + (\alpha_2, \alpha_2) = (\alpha_1 + \alpha_2,\ \alpha_1 + \alpha_2),\ (\alpha_1, \alpha_1)(x, x)(\alpha_2, \alpha_2) = (\alpha_1 x\alpha_2, \alpha_1 x\alpha_2)$

$\forall\, x,\ x_1, x_2 \in X$ and $\forall\, \alpha,\ \alpha_1, \alpha_2 \in \Gamma$. Then clearly X_1 is a Γ_1-ring.

Let $\emptyset_1 : X_1 \to X_1$ be an additive mapping defined by $\emptyset_1(x,x) = (\emptyset(x), \emptyset(x))$

Suppose $(x,\ x) = u$ and $(y,y) = v$, which are in X_1 and $(\alpha,\ \alpha) = \gamma$, which is in Γ_1.

Now, $\emptyset_1(u\gamma v) = \emptyset_1((x,\ x)(\alpha,\ \alpha)(y,y))$

$$
\begin{aligned}
&= \emptyset_1(x\alpha y,\ x\alpha y) \\
&= (\emptyset(x\alpha y),\ \emptyset(x\alpha y)) \\
&= (\emptyset(x)\alpha y + x\alpha\emptyset(y),\ \emptyset(x)\alpha y + x\alpha\emptyset(y)) \quad | \text{ since, } \emptyset \text{ is a derivation} \\
&= (\emptyset(x)\alpha y,\ \emptyset(x)\alpha y) + (x\alpha\emptyset(y),\ x\alpha\emptyset(y)) \\
&= \big(\emptyset(x), \emptyset(x)\big)(\alpha,\ \alpha)(y,y) + (x,\ x)(\alpha,\ \alpha)(\emptyset(y), \emptyset(y)) \\
&= \emptyset_1(x,x)\gamma v + u\gamma\emptyset_1(y,y) \\
&= \emptyset_1(u)\gamma v + u\gamma\emptyset_1(v)
\end{aligned}
$$

Hence, it follows that $\emptyset_1$ is a derivation.

1.6Definition: Let X be a Γ −ring. An additive mapping $\emptyset$: X→X is called an Inner derivation, if there exists an element $a \in X$ such that $\emptyset(\mathrm{x}) = a\alpha\mathrm{x} - \mathrm{x}\alpha a$, $\forall\ \mathrm{x} \in \mathrm{X}$, $\forall \alpha \in \Gamma$

Example 4: Let $X = \{\ \mathrm{x} = (\mathrm{a}_1\ \ \mathrm{b}_1) \mid \mathrm{a}_1, \mathrm{b}_1 \in R\ \}$ and $\Gamma = \left\{\alpha = \begin{pmatrix}\mathrm{c}\\ \mathrm{d}\end{pmatrix} \middle|\ \mathrm{c}, \mathrm{d} \in R\ \right\}$

Let $\mathrm{u} = (\mathrm{a}\ \ \mathrm{b}) \in \mathrm{X}$ be fixed

Now, $\mathrm{u}\alpha\mathrm{x} - \mathrm{x}\alpha\mathrm{u} = (\mathrm{a}\ \ \mathrm{b})\begin{pmatrix}\mathrm{c}\\ \mathrm{d}\end{pmatrix}(\mathrm{a}_1\ \ \mathrm{b}_1) - (\mathrm{a}_1\ \ \mathrm{b}_1)\begin{pmatrix}\mathrm{c}\\ \mathrm{d}\end{pmatrix}(\mathrm{a}\ \ \mathrm{b})$

$$
\begin{aligned}
&= (\mathrm{ac} + \mathrm{bd})(\mathrm{a}_1\ \ \mathrm{b}_1) - (\mathrm{a}_1\mathrm{c} + \mathrm{b}_1\mathrm{d})(\mathrm{a}\ \ \mathrm{b}) \\
&= (\mathrm{aca}_1 + \mathrm{bda}_1\quad \mathrm{acb}_1 + \mathrm{bdb}_1) - (\mathrm{a}_1\mathrm{ca} + \mathrm{b}_1\mathrm{da}\quad \mathrm{a}_1\mathrm{cb} + \mathrm{b}_1\mathrm{db}) \\
&= (\mathrm{aca}_1 + \mathrm{bda}_1 - \mathrm{a}_1\mathrm{ca} + \mathrm{b}_1\mathrm{da}\quad \mathrm{acb}_1 + \mathrm{bdb}_1 - \mathrm{a}_1\mathrm{cb} + \mathrm{b}_1\mathrm{db}) \\
&= (\ \mathrm{d}(\mathrm{ba}_1 - \mathrm{b}_1\mathrm{a})\qquad \mathrm{c}(\mathrm{ab}_1 - \mathrm{a}_1\mathrm{b}))
\end{aligned}
$$

Fix $\mathrm{u} = (2\ \ 5) \in \mathrm{X}$, $\alpha = \begin{pmatrix}\mathrm{c}\\ \mathrm{d}\end{pmatrix} \in \Gamma$, $\mathrm{x} = (\mathrm{a}\ \ \mathrm{b}) \in \mathrm{X}$

Let us define an additive map $\emptyset$: X→X by $\emptyset(\mathrm{x}) = (\ \mathrm{d}(5\mathrm{a} - 2\mathrm{b})\quad \mathrm{c}(2\mathrm{b} - 5\mathrm{a})\ )$ Then $\emptyset$ is an inner derivation.

1.7 Definition: Let X be a Γ −ring. An additive mapping $\emptyset$: X→X is called a Jordan derivation on X if $\emptyset(\mathrm{x}\alpha\mathrm{x}) = \emptyset(\mathrm{x})\alpha\mathrm{x} + \mathrm{x}\alpha\emptyset(\mathrm{x})$ $\forall\ \mathrm{x} \in \mathrm{X}$, $\forall \alpha \in \Gamma$

1.8 Definition: Let X be a Γ −ring. An additive mapping f: X →X is called a generalised derivation of X if there exist a derivation $\emptyset$: X→X such that $f(\mathrm{x}\alpha\mathrm{y}) = f(\mathrm{x})\alpha\mathrm{y} + \mathrm{x}\alpha\emptyset(\mathrm{y})$ $\forall$ x, y $\in$ X, $\forall \alpha \in \Gamma$ and f: X →X is called a Jordan generalised derivation of X if there exist a derivation $\emptyset$: X→X such that $f(\mathrm{x}\alpha\mathrm{x}) = f(\mathrm{x})\alpha\mathrm{x} + \mathrm{x}\alpha\emptyset(\mathrm{x})$, $\forall\ \mathrm{x} \in \mathrm{X}$, $\forall \alpha \in \Gamma$

Clearly, every generalised derivation is a Jordan generalised derivation. The converse is not true in general.

1.9Definition: Let X be a Γ −ring, $\emptyset$: X→X and $k : \Gamma \to \Gamma$ be additive mappings. If

$$\emptyset(\mathrm{x}\alpha\mathrm{y}) = \emptyset(\mathrm{x})\alpha\mathrm{y} + \mathrm{x}k(\alpha)\mathrm{y} + \mathrm{x}\alpha\emptyset(\mathrm{y})$$

holds $\forall$ x, y $\in$ X, $\forall \alpha \in \Gamma$, then $\emptyset$ is called a k-derivation of X.

Example5 : Let X be a Γ −ring and let $a \in \mathrm{X}$ and $\gamma \in \Gamma$ be any two fixed elements. Define the additive mapping $\emptyset$: X→X and $k : \Gamma \to \Gamma$ by $\emptyset(\mathrm{x}) = a\gamma\mathrm{x}$, $\forall\ \mathrm{x} \in \mathrm{X}$ and $k(\alpha) = -\alpha a\gamma$, $\forall \alpha \in \Gamma$, respectively.

Now, $\emptyset(\mathrm{x}\alpha\mathrm{y}) = a\gamma(\mathrm{x}\alpha\mathrm{y}) = a\gamma\mathrm{x}\alpha\mathrm{y} - \mathrm{x}\alpha a\gamma\mathrm{y} + \mathrm{x}\alpha a\gamma\mathrm{y}$

$$= (a\gamma\mathrm{x})\alpha\mathrm{y} - \mathrm{x}(\alpha a\gamma)\mathrm{y} + \mathrm{x}\alpha(a\gamma\mathrm{y}) = \emptyset(\mathrm{x})\alpha\mathrm{y} + \mathrm{x}k(\alpha)\mathrm{y} + \mathrm{x}\alpha\emptyset(\mathrm{y})$$

Thus, $\emptyset$ is a k-derivation of X.

1.10Definition: Let X be a Γ −ring and let $\emptyset$: X→X and $k : \Gamma \to \Gamma$ be additive mappings. If

$$\emptyset(x\alpha x) = \emptyset(x)\alpha x + xk(\alpha)x + x\alpha\emptyset(x)$$

holds $\forall$ x $\in$ X, $\forall\alpha \in \Gamma$, then $\emptyset$ is called a Jordan k-derivation of X

2. Main Results: The following are important results in this fields ,[3,5,7,8,11] .

Theorem 2.1: Let X be a prime Γ −ring with characteristic not equal to 2 and I a non zero ideal of X. Let f: X →X be a generalised derivation of X, associated with a derivation $\emptyset$. If $f(x) = 0, \ \forall$ x $\in I$, then $f = 0$.

Proof: $\forall$ x, y $\in$ I and $\forall\,\alpha \in \Gamma$, we have xαy $\in I$

By assumption, $f(x\alpha y) = 0$

$$\Rightarrow f(x\alpha y) = f(x)\alpha y + x\alpha\emptyset(y) = 0$$

$$\Rightarrow x\alpha\emptyset(y) = 0 \text{ ……….. } \mathbf{(A)}$$

Let $z \in$ X, $\beta \in \Gamma$, then, y$\beta z \in I$

Using (A) we get, $y\beta z\alpha\emptyset(y) = 0$

Because X is a prime Γ −ring and I is a nonzero ideal, so, $\emptyset(y) = 0$, $\forall$ y $\in$ I

Hence, by hypothesis, $f(r\alpha y) = 0,$ $\forall$ y $\in$ I and $\alpha \in \Gamma$ and r $\in$ X

$$\Rightarrow f(r)\alpha y + r\alpha\emptyset(y) = 0$$

$$\Rightarrow f(r)\alpha y = 0 \text{ ……….. } \mathbf{(B)}$$

Let w $\in$ X, $\gamma \in \Gamma$, then wγy $\in I$

Using (B) we get $f(r)\alpha w\gamma y = 0$

Again, because I is a nonzero ideal and primness of X

We get f(r) = 0 $\forall\, r \in$ X

Which gives, f = 0

Theorem 2.2: Let I be a nonzero ideal of a prime Γ −ring X, $a \in$ X and f $\neq$ 0 is a generalized derivation of X, with associated nonzero derivation $\emptyset$, then if $a\alpha f(x) = 0, \ \forall$ x $\in I$ and $\alpha \in \Gamma$, then $a = 0$

Proof. For any x $\in I$, $r \in$ X and $\alpha, \ \beta \in \Gamma$

$$a\alpha f(x\beta r) = 0 \quad | \text{ since, } x\beta r \in X$$

$$\Rightarrow a\alpha[f(x)\beta r + x\beta\emptyset(r)] = 0$$

$$\Rightarrow a\alpha f(x)\beta r + a\alpha\, x\beta\emptyset(r) = 0$$

$$\Rightarrow a\alpha\, x\beta\emptyset(r) = 0$$

Since, X is a prime Γ −ring and I is a nonzero ideal of X and also $\emptyset \neq 0$

We get $a = 0$

2.3 Definition: A Γ −ring X is called n-torsion free if $nx = 0 \Rightarrow x = 0$, $\forall$ x $\in$ X

2.4 Theorem: Let f: X →X be a generalised derivation and $\emptyset$: X→X be a derivation on a Γ −ring X. Let x, y, z $\in$ X and $\alpha, \ \gamma \in \Gamma$, then

(i) $f(x\alpha y + y\alpha x) = f(x)\alpha y + f(y)\alpha x + x\alpha\emptyset(y) + y\alpha\emptyset(x)$

(ii) $f(x\alpha y\gamma x + x\gamma y\alpha x) = f(x)\alpha y\gamma x + f(x)\gamma y\alpha x + x\gamma\emptyset(y)\alpha x + x\alpha\emptyset(y)\gamma x + x\alpha y\gamma\emptyset(x) + x\gamma y\alpha\emptyset(x)$

(iii) In particular, if X is 2-torsion free, then $f(x\alpha y\alpha x) = f(x)\alpha y\alpha x + x\alpha\emptyset(y)\alpha x + x\alpha y\alpha\emptyset(x)$

(iv) $f(x\alpha y\alpha z + z\alpha y\alpha x) = f(x)\alpha y\alpha z + f(z)\alpha y\alpha x + x\alpha\emptyset(y)\alpha z + z\alpha\emptyset(y)\alpha x + x\alpha y\alpha\emptyset(z) + z\alpha y\alpha\emptyset(x)$

Proof. (i) We have

$$f((x+y)\alpha(x+y)) = f(x\alpha x + x\alpha y + y\alpha x + y\alpha y)$$

$$\Rightarrow\ f(x+y)\alpha(x+y) + (x+y)\alpha\emptyset(x+y) = f(x\alpha x) + f(x\alpha y) + f(y\alpha x) + f(y\alpha y)$$

| for additive property of f and f is a generalised derivation $\Rightarrow$

$$[f(x)+f(y)]\alpha(x+y) + (x+y)\alpha[\emptyset(x)+\emptyset(y)] = f(x)\alpha x + x\alpha\emptyset(x) + f(x\alpha y) + f(y\alpha x) + f(y)\alpha y + y\alpha\emptyset(y)$$

$$\Rightarrow\ f(x)\alpha x + f(x)\alpha y + f(y)\alpha x + f(y)\alpha y + x\alpha\emptyset(x) + x\alpha\emptyset(y) + y\alpha\emptyset(x) + y\alpha\emptyset(y) = f(x)\alpha x + x\alpha\emptyset(x) + f(x\alpha y) + f(y\alpha x) + f(y)\alpha y + y\alpha\emptyset(y)$$

$$\Rightarrow\ f(x)\alpha y + f(y)\alpha x + x\alpha\emptyset(y) + y\alpha\emptyset(x) = f(x\alpha y) + f(y\alpha x)$$

Hence, $f(x\alpha y + y\alpha x) = f(x)\alpha y + f(y)\alpha x + x\alpha\emptyset(y) + y\alpha\emptyset(x)$

(ii) Replacing y with $y\gamma x$ and $x\gamma y$ in (i), we get

$$f(x\alpha y\gamma x + x\gamma y\alpha x) = f(x)\alpha y\gamma x + f(x\gamma y)\alpha x + x\alpha\emptyset(y\gamma x) + x\gamma y\alpha\emptyset(x)$$

$$= f(x)\alpha y\gamma x + [f(x)\gamma y + x\gamma\emptyset(y)]\alpha x + x\alpha[\emptyset(y)\gamma x + y\gamma\emptyset(x)] + x\gamma y\alpha\emptyset(x)$$

Hence, $f(x\alpha y\gamma x + x\gamma y\alpha x) = f(x)\alpha y\gamma x + f(x)\gamma y\alpha x + x\gamma\emptyset(y)\alpha x + x\alpha\emptyset(y)\gamma x + x\alpha y\gamma\emptyset(x) + x\gamma y\alpha\emptyset(x)$

(iii) Putting $\gamma = \alpha$ in (ii), we get

, $$f(x\alpha y\alpha x + x\alpha y\alpha x) = f(x)\alpha y\alpha x + f(x)\alpha y\alpha x + x\alpha\emptyset(y)\alpha x + x\alpha\emptyset(y)\alpha x + x\alpha y\alpha\emptyset(x) + x\alpha y\alpha\emptyset(x) \qquad (*)$$

$$\Rightarrow\ f(2x\alpha y\alpha x) = 2f(x)\alpha y\alpha x + 2x\alpha\emptyset(y)\alpha x + 2x\alpha y\alpha\emptyset(x)$$

Since, X is 2-torsion free

Therefore, we get

$$f(x\alpha y\alpha x) = f(x)\alpha y\alpha x + x\alpha\emptyset(y)\alpha x + x\alpha y\alpha\emptyset(x)$$

(iv) Replacing x with z in (*) in (iii) , we get the result (iv)

2.5 Theorem: Let $\emptyset$: X→X be a Jordan k-derivation on a Γ −ring X. Then ∀ x, y, z ∈ X and ∀ $\alpha,\ \gamma \in \Gamma$, the following statements hold:

(i) $\emptyset(x\alpha y + y\alpha x) = \emptyset(x)\alpha y + \emptyset(y)\alpha x + xk(\alpha)y + yk(\alpha)x + x\alpha\emptyset(y) + y\alpha\emptyset(x)$

(ii) $\emptyset(x\alpha y\gamma x + x\gamma y\alpha x) = \emptyset(x)\alpha y\gamma x + \emptyset(x)\gamma y\alpha x + xk(\alpha)y\gamma x + xk(\gamma)y\alpha x + x\gamma\emptyset(y)\alpha x + x\alpha\emptyset(y)\gamma x + x\alpha yk(\gamma)x + x\gamma yk(\alpha)x + x\alpha y\gamma\emptyset(x) + x\gamma y\alpha\emptyset(x)$

(iii) In particular, if X is 2-torsion free, then

$$\emptyset(x\alpha y\alpha x) = \emptyset(x)\alpha y\alpha x + xk(\alpha)y\alpha x + x\alpha\emptyset(y)\alpha x + x\alpha yk(\alpha)x + x\alpha y\alpha\emptyset(x)$$

(iv) $\emptyset(x\alpha y\alpha z + z\alpha y\alpha x) = \emptyset(x)\alpha y\alpha z + \emptyset(z)\alpha y\alpha x + xk(\alpha)y\alpha z + zk(\alpha)y\alpha x + x\alpha\emptyset(y)\alpha z + z\alpha\emptyset(y)\alpha x + x\alpha yk(\alpha)z + z\alpha yk(\alpha)x + x\alpha y\alpha\emptyset(z) + z\alpha y\alpha\emptyset(x)$

Proof: (i) We have

$$\emptyset((x+y)\alpha(x+y)) = \emptyset(x\alpha x + x\alpha y + y\alpha x + y\alpha y)$$

$\Rightarrow$ $\emptyset(x+y)\alpha(x+y) + (x+y)k(\alpha)(x+y) + (x+y)\alpha\emptyset(x+y) = \emptyset(x\alpha x) + \emptyset(x\alpha y) + \emptyset(y\alpha x) + \emptyset(y\alpha y)$

| for additive property of $\emptyset$ and $\emptyset$ is a Jordan k-derivation

$\Rightarrow$ $[\emptyset(x)+\emptyset(y)]\alpha(x+y) + xk(\alpha)x + xk(\alpha)y + yk(\alpha)x + yk(\alpha)y + (x+y)\alpha[\emptyset(x) + \emptyset(y)] = \emptyset(x)\alpha x + xk(\alpha)x + x\alpha\emptyset(x) + \emptyset(x\alpha y) + \emptyset\ (y\alpha x) + \emptyset(y)\alpha y + yk(\alpha)y + y\alpha\emptyset(y)$

$\Rightarrow$ $\emptyset(x)\alpha x + \emptyset(x)\alpha y + \emptyset(y)\alpha x + \emptyset(y)\alpha y + xk(\alpha)x + xk(\alpha)y + yk(\alpha)x + yk(\alpha)y + x\alpha\emptyset(x) + x\alpha\emptyset(y) + y\alpha\emptyset(x) + y\alpha\emptyset(y) = \emptyset(x)\alpha x + xk(\alpha)x + x\alpha\emptyset(x) + \emptyset(x\alpha y + y\alpha x) + \emptyset(y)\alpha y + yk(\alpha)y + y\alpha\emptyset(y)$

Hence, $\emptyset(x\alpha y + y\alpha x) = \emptyset(x)\alpha y + \emptyset(y)\alpha x + xk(\alpha)y + yk(\alpha)x + x\alpha\emptyset(y) + y\alpha\emptyset(x)$

(ii) Replacing y with $y\gamma x$ and $x\gamma y$ in (i), we get

$$\emptyset(x\alpha y\gamma x + x\gamma y\alpha x) = \emptyset(x)\alpha y\gamma x + \emptyset(x\gamma y)\alpha x + xk(\alpha)y\gamma x + x\gamma yk(\alpha)x + x\alpha\emptyset(y\gamma x) + x\gamma y\alpha\emptyset(x)$$

$\Rightarrow$ $\emptyset(x\alpha y\gamma x + x\gamma y\alpha x) = \emptyset(x)\alpha y\gamma x + \emptyset(x\gamma y)\alpha x + xk(\alpha)y\gamma x + x\gamma yk(\alpha)x + x\alpha\emptyset(y\gamma x) + x\gamma y\alpha\emptyset(x)$

$= \emptyset(x)\alpha y\gamma x + [\emptyset(x)\gamma y + xk(\gamma)y + x\gamma\emptyset(y)]\alpha x + xk(\alpha)y\gamma x + x\gamma yk(\alpha)x + x\alpha[\ \emptyset(y)\gamma x + yk(\gamma)x + y\gamma\emptyset(x)] + x\gamma y\alpha\emptyset(x)$

After simplification, we get

$$\emptyset(x\alpha y\gamma x + x\gamma y\alpha x) = \emptyset(x)\alpha y\gamma x + \emptyset(x)\gamma y\alpha x + xk(\alpha)y\gamma x + xk(\gamma)y\alpha x + x\gamma\emptyset(y)\alpha x + x\alpha\emptyset(y)\gamma x + x\alpha yk(\gamma)x + x\gamma yk(\alpha)x + x\alpha y\gamma\emptyset(x) + x\gamma y\alpha\emptyset(x)$$

(iii) Putting $\gamma = \alpha$ in (ii), we get

$$\emptyset(x\alpha y\alpha x + x\alpha y\alpha x) = \emptyset(x)\alpha y\alpha x + \emptyset(x)\alpha y\alpha x + xk(\alpha)y\alpha x + xk(\alpha)y\alpha x + x\alpha\emptyset(y)\alpha x + x\alpha\emptyset(y)\alpha x + x\alpha yk(\alpha)x + x\alpha yk(\alpha)x + x\alpha y\alpha\emptyset(x) + x\alpha y\alpha\emptyset(x) \quad \ldots\ldots\ldots(*)$$

$$\emptyset(2x\alpha y\alpha x) = 2\emptyset(x)\alpha y\alpha x + 2xk(\alpha)y\alpha x + 2x\alpha\emptyset(y)\alpha x + 2x\alpha yk(\alpha)x + 2x\alpha y\alpha\emptyset(x)$$

Since, X is 2-torsion free

Therefore, we get

$$\emptyset(x\alpha y\alpha x) = \emptyset(x)\alpha y\alpha x + xk(\alpha)y\alpha x + x\alpha\emptyset(y)\alpha x + x\alpha yk(\alpha)x + x\alpha y\alpha\emptyset(x) \qquad (iv)$$

If we replace x by z in (*), then we get (iv)

2.6 Definition: Let $\emptyset : X \rightarrow X$ be a Jordan k-derivation on a Γ −ring X. $\forall$ x, y $\in$ X and $\forall\, \alpha \in \Gamma$, we define $f_\alpha(x,y) = \emptyset(x\alpha y) - \emptyset(x)\alpha y - xk(\alpha)y - x\alpha\emptyset(y)$

2.7 Theorem: If $\emptyset : X \rightarrow X$ be a Jordan k-derivation on a Γ −ring X. Then $\forall$ x, y, z $\in$ X and $\forall\, \alpha,\ \gamma \in \Gamma$, the following statements hold:

(a) $f_\alpha(x,y) + f_\alpha(y,x) = 0$

(b) $f_\alpha(x+\ y,z) = f_\alpha(x,z) + f_\alpha(y,z)$

(c) $f_\alpha(x,y+z) = f_\alpha(x,z) + f_\alpha(y,z)$

(d) $f_{\alpha+\gamma}(x,y) = f_\alpha(x,y) + f_\gamma(x,y)$

Proof. (a) We have

$$f_{\alpha}(x,y) + f_{\alpha}(y,x) = \emptyset(x\alpha y) - \emptyset(x)\alpha y - xk(\alpha)y - x\alpha\emptyset(y) + \emptyset(y\alpha x) - \emptyset(y)\alpha x - yk(\alpha)x - y\alpha\emptyset(x)$$

$$= \emptyset(x)\alpha y + xk(\alpha)y + x\alpha\emptyset(y) - \emptyset(x)\alpha y - xk(\alpha)y - x\alpha\emptyset(y) + \emptyset(y)\alpha x + yk(\alpha)x + y\alpha\emptyset(x) - \emptyset(y)\alpha x - yk(\alpha)x - y\alpha\emptyset(x)$$

$$= 0$$

(b) $f_{\alpha}(x + y, z) = \emptyset((x+y)\alpha z) - \emptyset(x+y)\alpha z - (x+y)k(\alpha)z - (x+y)\alpha\emptyset(z)$

$$= \emptyset(x\alpha z + y\alpha z) - (\emptyset(x) + \emptyset(y))\alpha z - xk(\alpha)z - yk(\alpha)z - x\alpha\emptyset(z) - y\alpha\emptyset(z)$$

$$= \emptyset(x\alpha z) + \emptyset(y\alpha z) - \emptyset(x)\alpha z - \emptyset(y)\alpha z - xk(\alpha)z - yk(\alpha)z - x\alpha\emptyset(z) - y\alpha\emptyset(z)$$

$$= \{\emptyset(x\alpha z) - \emptyset(x)\alpha z - xk(\alpha)z - x\alpha\emptyset(z)\} + \{\emptyset(y\alpha z) - \emptyset(y)\alpha z - yk(\alpha)z - y\alpha\emptyset(z)\}$$

$$= f_{\alpha}(x,z) + f_{\alpha}(y,z)$$

(c) It is similar to (b)

(d) We have

$$f_{\alpha+\gamma}(x,y) = \emptyset(x(\alpha+\gamma)y) - \emptyset(x)(\alpha+\gamma)y - xk(\alpha+\gamma)y - x(\alpha+\gamma)\emptyset(y)$$

$$= \emptyset(x\alpha y) + \emptyset(x\gamma y) - \emptyset(x)\alpha y - \emptyset(x)\gamma y - xk(\alpha)y - xk(\gamma)y - x\alpha\emptyset(y) - x\gamma\emptyset(y)$$

$$= \{\emptyset(x\alpha y) - \emptyset(x)\alpha y - xk(\alpha)y - x\alpha\emptyset(y)\} + \{\emptyset(x\gamma y) - \emptyset(x)\gamma y - xk(\gamma)y - x\gamma\emptyset(y)\}$$

$$= f_{\alpha}(x,y) + f_{\gamma}(x,y)$$

Problem: Can we extend these results to the Projective tensor product of n-number of Γ −rings?

Acknowledgement : We are very much grateful to Dr Tarini Kumar Dutta, Professor of Mathematics, Assam Don Bosco University, Guwahati , Assam State .

References

[1] N. Nobusawa, Structure theorems of a module over a ring with bilinear mapping, Canad. Math. Bull., 10 (1967), pp. 649-652.

[2] W. E. Barnes, On the Γ-rings of Nobusawa, Pacific J. Math., 18 (1966), pp. 411-422.

[3] Mehmet Ali Ozturk, Young Bae Jun and Kyung Ho Kim, On Derivations of Prime Gamma Rings, Turk J Math. , 26(2002), pp. 317-327.

[4] C. Jaya Subba Reddy, K. Nagesh and A. Sivakameshwara Kumar, Left Generalized Derivations on Prime Γ −rings, Annals of Pure and Applied Mathematics, Vol. 16, No. 1, 2018, pp. 127 – 131.

[5] A.R Khan, M. Anwar Chaudhry and Imran Javaid, generalized derivations on prime Γ −rings, World Applied Sciences Journal, 23(12) (2013) pp. 59-64.

[6] S. Kyuno, Prime ideals in gamma-rings, Pacific J. Math., 98 (1982), pp. 375-379

[7] J. Luh, On the theory of Simple Γ- rings, Michigan Math. J., 16 (1969), pp. 65-75.

[8] W.G. Lister, Ternary rings, Trans. Amer. Math. Soc., 154 (1971), pp. 37-55.

[9] S. Chakraborty and A.C Paul, Jordan k-derivations of certain Nobusawa Gamma rings, GANIT J. Bangladesh Math. Soc., 31 (2011) 53-64.

[10] Y. Ceven and M. Ali Ozturk, On Jordan generalized derivations in Gamma rings, Hacettepe Journal of Mathematics and Statistics, 33 (2024) 11-14.

[11] N Herstein, Topics in Ring Theory, University of Chicago Press, Chicago, 1969.