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Connected Domination number of a Commutative Ring

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Abstract

In this paper, we evaluate the connected domination number of $\Gamma(Z_n)$, in some case of n. We find out that the connected domination number of $\Gamma(Z_{p_1^{e_1}\times p_2^{e_2}\times\cdots\times p_k^{e_k}})$ is equal to k. Finally, we characterize the graphs in which $\gamma(\Gamma(Z_n)) = \gamma_c(\Gamma(Z_n))$.

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1. Introduction

Let R be a commutative ring and let Z(R) be its set of zero-divisors. We associate a graph $\Gamma(R)$ to R with vertices $Z(R)^* = Z(R) - \{0\}$, the set of non-zero zero divisors of R and for distinct $x, y \in Z(R)^*$, the vertices x and y are adjacent if and only if $xy = 0$. Thus, $\Gamma(R)$ is the empty graph if and only if R is an integral Domain. Throughout this paper, we consider the commutative ring R by Z_n and zero divisor graph $\Gamma(R)$ by $\Gamma(Z_n)$.The idea of a zero-divisor graph of a commutative ring was introduced by I. Beck in [2], where he was mainly interested in colorings. The zero divisor graph is very useful to find the algebraic structures and properties of rings [1].

Let graph G=(V, E) be a graph of order n. A set D $\subseteq$ V is a dominating set if every vertex in V-D is adjacent to atleast one vertex in D. The domination number $\gamma(\Gamma(Z_n))$

is the minimum cardinality of a dominating set of $\Gamma(Z_n)$. The private neighbor set of a vertex v with respect to a set D, denoted by $pn[v, D]$ is $N[v] - N[D - \{v\}]$ and each $u \in pn[v, D]$ is called a private neighbor of v with respect to D. A connected domination set D is a set of vertices of a graph G such that every vertex in V-D is adjacent to atleast one vertex in D and the subgraph $< D >$ induced by the set D is connected. The connected domination number $\gamma_c(G)$ is the minimum of the cardinalities of the connected dominating sets of G.

Claude Berge in his book [3] defined for the first time the concept of the domination number of a graph. An elabourate treatment of domination parameter appears in Cockayene and Hedetneimi [4]. The term connected domination set was first suggested by S.T. Hedetniemi and elabourate treatment of this parameter appears in E. Sampathkumar and H.B. Walikar [5].

2. Preliminaries

Lemma 2.1. A graph G has a connected domination set iff G is connected [5].

Lemma 2.2. A subset D of $V(\Gamma(Z_n))$ is a connected domination set iff $\Gamma(Z_n)$ has a spanning tree T satisfying the following conditions;

(a) Each $v \in V(\Gamma(Z_n)) - D$ is a pendent vertex in T.

(b) For every subset $S \subseteq V(\Gamma(Z_n)) - D$ with $< S >$ independent in G, there exists a non pendent vertex v in T such that $S \subseteq N(v)$.

Lemma 2.3. A graph $\Gamma(Z_n)$ has a connected domination set iff $\Gamma(Z_n)$ is connected and n is a composite number.

Proof. Let $\Gamma(Z_n)$ be a graph with connected domination. Then $< S >$ is connected and every $x \in V(\Gamma(Z_n)) - S$ is adjacent to some $y \in S$. Clearly, $\Gamma(Z_n)$ is connected.

Conversely, let $\Gamma(Z_n)$ be a connected graph then the following conditions are holds,

(a) If $\Gamma(Z_n)$ is a block then $S = V - \{u\}$ is a connected domination set, for any $u \in V(\Gamma(Z_n))$.

(b) If $\Gamma(Z_n)$ is a separable graph then $S = V(\Gamma(Z_n)) - \{u\}$ is a connected domination set for any non cut vertex $u \in V(G)$. Hence, every connected graph $\Gamma(Z_n)$ has a connected domination set.

(c) If n is prime, then $\Gamma(Z_n)$ is an integral domain and it has no zero divisor. Hence, n is a composite number. ■

Remarks 2.4.

(a) If $\Gamma(Z_n)$ is a tree with $v \in V(\Gamma(Z_n))$ is a support and if A_v denotes the set of all pendant vertices at v, then $D = V(\Gamma(Z_n)) - A_v$ is a connected domination set of $\Gamma(Z_n)$.

(b) $Av \leq \Delta$, $Av \leq \in$, where $\in$ denote the number of pendent vertices in a spaning tree with maximum number of pendent edges.

(c) $\gamma_c(P_2) = 1, \gamma_c(P_3) = 2$.

(d) $\gamma_c(C_n) = n - 2$, for every positive integer n.

(e) $\gamma_c(K_n) = 1, \gamma_c(K_{m,n}) = 2$, for every positive integer n and m.

3. Connected Domination Number Of $\Gamma(Z_n)$

In this section, we compute the Connected Domination Number of $\Gamma(Z_n)$.

Theorem 3.1. For $\Gamma(Z_{2p})$, where p is any prime number then $\gamma_c(\Gamma(Z_{2p})) = 1$. Also, if $n = 8, 9$ then $\gamma_c(\Gamma(Z_n)) = 1$.

Proof. The vertex set of $\Gamma(Z_{2p})$ is $\{2, 4, 6, 2(p-1), p\}$. Let $u = 2(p-1)$ and $v = p$ then $uv = 2(p-1).p = 2p(p-1)$. Clearly, $2p$ must divides $2p(p-1)$, then there exist a edge connect between u and v. Similarly, let u be any vertex in $\{2, 4, 6, \ldots, 2(p-1)\}$ and $v = p$ then $2p$ must divides uv. Note that, v is adjacent to all the vertices in $\Gamma(Z_{2p})$ and hence $\gamma_c(\Gamma(Z_{2p})) = 1$.

If $n = 8$, then the vertex set of $\Gamma(Z_n)$ is $\{2, 4, 6\}$, then the vertex 4 is adjacent to 2 and 6. That is 2.4=0 and 4.6=0. Thus $\gamma_c(\Gamma(Z_n)) = 1$. Similarly, if $n = 9$, then the vertex set of $\Gamma(Z_n)$ is $\{3, 6\}$ and hence $\gamma_c(\Gamma(Z_n)) = 1$. ■

Theorem 3.2. For any graph $\Gamma(Z_{2p})$ with p vertices and maximum vertex degree $\Delta(\Gamma(Z_{2p}))$ then $\gamma_c(\Gamma(Z_{2p})) = p - \Delta(\Gamma(Z_{2p}))$, if and only if $\Gamma(Z_{2p})$ is a star graph.

Proof. Let v be a vertex with maximum degree $\Delta(\Gamma(Z_{2p}))$. If $\Gamma(Z_{2p})$ is a star with v as the root, then the graph $\Gamma(Z_{2p})$ has exactly $\Delta(\Gamma(Z_{2p}))$ branches from v. Since, the vertices in each of these branches has a degree less than 3. Thus the number of leaves in $\Gamma(Z_{2p})$ is exactly $\Delta(\Gamma(Z_{2p}))$. Using theorem (3.1), the connected domination number of $\Gamma(Z_{2p})$ is 1. That is, 1 = number of points - maximum degree = $p - (p-1) = 1$ and hence, $\gamma_c(\Gamma(Z_{2p})) = p - \Delta(\Gamma(Z_{2p}))$.

Conversely, if $\Gamma(Z_{2p})$ is not a star, then there exists a vertex other that v with degree not less than 3 in $\Gamma(Z_{2p})$. Therefore, $\Gamma(Z_{2p})$ has a branch with more that one leaf in it. This shows that $\Gamma(Z_{2p})$ has more than $\Delta(\Gamma(Z_{2p}))$ leaves, which is a contradiction and hence the theorem. ■

Theorem 3.3. In $\Gamma(Z_{3p})$ where p is any prime with > 3, then $\gamma_c(\Gamma(Z_{3p})) = 2$.

Proof. The vertex set of $\Gamma(Z_{3p})$ is $\{3, 6, 9, ..., 3(p-1), p, 2p\}$. Let a vertex $v \in \Gamma(Z_{3p})$ with $deg(v) = \Delta$. Suppose u be another vertex with $deg(u) = \Delta$ in $\Gamma(Z_{3p})$, then either $u = p, v = 2p$ or $u = 2p, v = p$.

Then $uv = 2p \times p = 2p^2$ which does not divide by $3p$. Therefore u and v are non adjacent vertices in $\Gamma(Z_{3p})$. Let w be any other vertex in $\Gamma(Z_{3p})$ such that $uw = vw = 0$.

That is the remaining vertices in $\Gamma(Z_{3p})$ are adjacent to both u and v. Clearly, the connected domination set $D = \{u, w\}$ or $D = \{v, w\}$ and hence, $\gamma_c(\Gamma(Z_{3p})) = 2$. ■

Theorem 3.4. For any prime $p \geq 5$, then $\gamma_c(\Gamma(Z_{4p})) = 2$.

Proof. The vertex set of $\Gamma(Z_{4p})$ is $\{2, 4, 6, \dots, 2(2p-1), p, 2p, 3p\}$. Let $u = 2p$ and v is any even number from 2 to $2(2p-1)$. Clearly, $uv = 2p \times 2(2p-1) = 4p(p-1) = (p-1)(0) = 0$. That is, $4p$ must divides uv, then u and v are adjacent. Also note that, let $u = 2p$, $v = p$ and $w = 3p$ then, $uv = 2p.p$, $uw = 2p.3p$, which implies that $4p$ does not divides $uv = 2p^2$ and $uw = 6p^2$. So, u, v and w are non adjacent vertices in $\Gamma(Z_{4p})$.

Let a vertex $x = 4$ in $\Gamma(Z_{4p})$, then $4p$ must divides $xv = 4.p$ and $xw = 4.p^2$. That is x is adjacent to both v and w. Clearly, the connected domination set $D = \{x, u\} = \{4, 2p\}$. Hence, $\gamma_c(\Gamma(Z_{4p})) = 2$. ■

Theorem 3.5. If $p > 5$ is any prime, then $\gamma_c(\Gamma(Z_{5p})) = 2$.

Proof. Let v be any vertex with maximum degree. The vertex set of $\Gamma(Z_{5p})$ is $\{5, 10, \dots, 5(p-1), p, 2p, \dots, 4p\}$. Clearly, the vertex V can be partition in the two parts V_1 and V_2. That is, $V_1 = \{5, 10, \dots, 5(p-1)\}$ and $V_2 = \{p, 2p, 3p, 4p\}$.

Let, $u = 5$ and $v = 10$ in V_1, then $5p$ does not divides 50. Note that in V_2, $u = 2p$ and $v = 3p$ then $5p$ does not divides $uv = 6p^2$, which implies that no two vertices of V_1 and V_2 are adjacent.

Let x is any vertex in V_1, say $x = 10$ and y is any vertex in V_2, say $y = 2p$. Then, $xy = 10 \times 2p = 20p$. Clearly, $5p$ must divides $20p$. That is, x and y are adjacent in $\Gamma(Z_{5p})$. Using the same process, finally we get, every vertex in V_1 is adjacent to all the vertices in V_2 and D={Any one of the vertex in V_1, Any one of the vertex in V_2} and hence $\gamma_c(\Gamma(Z_{5p})) = 2$. ■

Theorem 3.6. For any graph $\Gamma(Z_{7p})$ where p is any prime > 7, then, $\gamma_c(\Gamma(Z_{7p})) = 2$.

Proof. The vertex set of $\Gamma(Z_{7p})$ is $\{7, 14, \dots, 7(p-1), p, 2p, \dots, 6p\}$. Let u be any vertex, say 7 and v be any vertex, say p then $7p$ must divide uv, which implies that u and v are adjacent vertices in $\Gamma(Z_{7p})$.

Let $x = 7$ and $y = 14$ in $\Gamma(Z_{7p})$ then $7p$ does not divide xy. That is $7p$ does not divide 84. It seems that, the vertex set of $\Gamma(Z_{7p})$ partition in the two parts say V_1 and V_2. Clearly any vertex in V_1 is adjacent to all the vertices in V_2, similarly any vertex in V_2 is adjacent to all the vertices in V_1. That is the connected domination set D is {Any one vertex from V_1, Any one of the vertex in V_2} and hence, $\gamma_c(\Gamma(Z_{7p})) = 2$. ■

Theorem 3.7. If p and q are distinct prime and $q > p$, then $\gamma_c(\Gamma(Z_{pq})) = 2$.

Proof. Using theorem (3.6) and (3.7), we get $\gamma_c(\Gamma(Z_{5p})) = \gamma_c(\Gamma(Z_{7p})) = 2$. Similarly, we get $\gamma_c(\Gamma(Z_{11p})) = \gamma_c(\Gamma(Z_{13p})) = 2$, where $p > 11$ and $p > 13$ respectively. Continue the same process, finally we get $\gamma_c(\Gamma(Z_{pq})) = 2$. ■

Theorem 3.8. If p and q are distinct prime and n is a positive integer greater that one, then $\gamma_c(\Gamma(Z_{p^n q})) = 2$.

Proof. Using [6], $\Gamma(Z_{p^n q})$ can be partition into $p^{n/2}$ if n is even and $(p^{(n-1)/2} + 1)$, if n is odd.

In $\Gamma(Z_{p^n q})$, we can find four vertices defined as $x_1 = p$, $x_2 = p^{n-1}q$, $x_3 = p^n$, $x_4 = q$. Clearly, $x_1x_2 = x_2x_3 = x_3x_4 = 0$, but $x_2x_4 \neq 0$ and $x_1x_4 \neq 0$. That is $p^n q$ does not divide x_2x_4 which implies $p^n q$ does not divide $p^{n-1}q^2$ and same as x_1x_4. Therefore diameter of $\Gamma(Z_{p^n q}) = 3$.

Clearly, there exist two vertices in $\Gamma(Z_{p^n q})$ are covers remaining all vertices in $\Gamma(Z_{p^n q})$ and hence, $\gamma_c(\Gamma(Z_{p^n q})) = 2$. ■

Theorem 3.9. If p is any prime then $\gamma_c(\Gamma(Z_{p^2})) = 1$.

Proof. The vertex set of $\Gamma(Z_{p^2})$ is $\{p, 2p, 3p, \dots, (p-1)\}$. Clearly, p is adjacent to all the vertices in $V(\Gamma(Z_{p^2}))$. Also note that any two vertices in $\Gamma(Z_{p^2})$ is adjacent and hence, $\gamma_c(\Gamma(Z_{p^2})) = 1$. ■

Theorem 3.10. For any graph $\Gamma(Z_{2^n})$ where $n > 3$, then $\gamma_c(\Gamma(Z_{2^n})) = 1$.

Proof. Let $v \in \Gamma(Z_{2^n})$ has a maximum degree Δ which implies that $deg(v) = 2^{n-1} - 2$. The vertex set of $\Gamma(Z_{2^n})$ is $\{2, 4, 6, \dots, 2^{n-1}, 2(2^{n-1} - 1)\}$. Let $v = 2^{n-1}$ and w be any other vertex in $\Gamma(Z_{2^n})$. Suppose $w = 2^n - 2$, then $vw = (2^{n-1}) \times (2^n - 2) = 2^{n+(n-1)} - 2^n = 2^n(2^{n-1} - 1)$. Clearly, 2^n must divides $2^n(2^{n-1} - 1)$. Thus, the vertex v is adjacent to all vertices in $\Gamma(Z_{2^n})$ and hence, $\gamma_c(\Gamma(Z_{2^n})) = 1$. ■

Theorem 3.11. In $\Gamma(Z_{3^n})$, where $n \geq 3$, then $\gamma_c(\Gamma(Z_{3^n})) = 1$.

Proof. Since, $\Gamma(Z_{3^n})$ has no pendent vertex and there exists two vertices u and v are adjacent to all the vertices in $\Gamma(Z_{3^n})$. That is there exists any vertex $w \in V(\Gamma(Z_{3^n}))$, such that w is adjacent to both u and v.

The vertex set of $\Gamma(Z_{3^n})$ is $\{3, 6, 9, \dots, 3^{n-1}, \dots, 2.3^{n-1}, \dots, 3.(3^{n-1} - 1)\}$. Let $u = 3^{n-1}$ and $v = 2 \times 3^{n-1}$, then $uv = 2.3^{2(n-1)} = 3^n(2 \times 3^{n-2})$ and 3^n must divide $3^n(2 \times 3^{n-2})$, then there exists an edge connect between u and v. Clearly, the connected domination set $D = \{u\}$ or $\{v\}$ and hence, $\gamma_c(\Gamma(Z_n)) = 1$. ■

Theorem 3.12. If p is any prime, then $\gamma_c(\Gamma(Z_{p^n})) = 1$.

Proof. Using theorem (3.4) and (3.5), if $p = 2$ or $p = 3$, then $\gamma_c(\Gamma(Z_{p^n})) = 1$. In general, there exists a vertex v in $\Gamma(Z_{p^n})$ is adjacent to all vertices in $\Gamma(Z_{p^n})$ and hence, $\gamma_c(\Gamma(Z_{p^n})) = 1$. ■

4. Main Results

In this section, we find out that the connected domination number of

$$\Gamma(Z_{p_1^{e_1} \times p_2^{e_2} \times \cdots \times p_k^{e_k}})$$

is equal to k. Finally, we characterize the graphs in which $\gamma(\Gamma(Z_n)) = \gamma_c(\Gamma(Z_n))$.

Theorem 4.1. For any graph $\Gamma(Z_n)$, if $n = p^n q^m$, where p and q are prime numbers and n, m are positive integers with $n \geq 2, m \geq 1$ then $\gamma_c(\Gamma(Z_n)) = 2$.

Proof. The vertex set in $\Gamma(Z_{p^n q^m})$ is $\{2, 4, \ldots, (pqr - 2), 3, 6, \ldots, (pqr - 3),\ p, q, r, pq, qr, pr\}$.

Let $x, y \in V(\Gamma(Z_n))$, then x/n or y/n or n/xy. Clearly $xy = 0$ and there exist an edge connect between x and y. Since $(x, y) \neq 1$ and there exist any vertex $z \in V(\Gamma(Z_n))$ either n/xz or n/yz then, $xz = 0$ or $yz = 0$. Thus, every vertex in $\Gamma(Z_n)$ is adjacent to either x or y. Using theorem (3.8) and [6], $\gamma_c(\Gamma(Z_n)) = 2$. ■

Theorem 4.2. Let $n = p^n q^m r^k$, where p, q, r are distinct primes and n, m, k are positive integers with $n, m, k \geq 1$, then $\gamma_c(\Gamma(Z_n)) = 3$.

Proof. Let $x = pq$, $y = qr$ and $z = pr$ in $V(\Gamma(Z_n))$. Then, $xy = pq.qr = pr^2r$, $yz = qr.pr = pqr^2$ and $xz = pq.pr = p^2qr$ implies that $n/xy, n/yz$ and n/xz. The vertices x, y and z are adjacent, and the graph $\Gamma(Z_n)$ has a K_3 subgraph. Clearly $(x, y, z) = 1$. That is x, y and z are relatively prime numbers. Similarly (y,z) and (x,z). Let v be any other vertex in $\Gamma(Z_n)$ then $xv = 0$ or $yv = 0$. It mean that v is adjacent to any one of the vertex from $\{x, y, z\}$. Clearly, {x,y,z} covers all the vertices in $\Gamma(Z_n)$, and hence $\gamma_c(\Gamma(Z_n)) = 3$. ■

Theorem 4.3. Let n = $p_1^{e_1} p_2^{e_2}, \ldots, p_k^{e_k}$, where $p_1, p_2, \ldots, p_k$ are distinct primes and the $e_i's$ are positive integers, then $\gamma_c(\Gamma(Z_n)) = k$.

Proof. Using theorem (4.1), we get $\gamma_c(\Gamma(Z_{p^n q^m})) = 2$ and using theorem (4.2), we get $\gamma_c(\Gamma(Z_{p^n q^m r^k})) = 3$. Similarly, proceeding the same way, Finally we get $\Gamma(Z_n)$ has a subgraph of K_k.

Let v be any other vertex in $\Gamma(Z_n)$ then any one of the following is true. (a) $x_1 v = 0$ or (b) $x_2 v = 0$ or .... (k) $x_k v = 0$. That is, remaining vertices in $\Gamma(Z_n)$ is adjacnet to any one of vertex in $K_k = \{x_1, x_2, \ldots, x_k\}$ and hence, $\gamma_c(\Gamma(Z_n)) = k$. ■

Theorem 4.4. For any graph $\Gamma(Z_{2p})$, $\gamma(\Gamma(Z_{2p})) = \gamma_c(\Gamma(Z_{2p}))$ iff $\Gamma(Z_{2p})$ is a star.

Proof. Let $\gamma(\Gamma(Z_{2p})) = \gamma_c(\Gamma(Z_{2p}))$ and S be a γ_c set of $\Gamma(Z_{2p})$. Then S=V($\Gamma(Z_{2p})$) - q, where q is the number of end points of $\Gamma(Z_{2p})$ which implies that,

$$|V(\Gamma(Z_{2p})) - q| = |S| = \gamma_c(\Gamma(Z_{2p})) = \gamma(\Gamma(Z_{2p})) \leq p - \Delta$$

where p is number of points and Δ is maximum degree. Therefore $q \geq \Delta$. Using Theorem (3.1) and (3.2), we get $q = \Delta$. Using theorem (3.1), $D = \{p\}$ and $N(p) \cap D =$

ϕ. That is $V(\Gamma(Z_{2p})) - D = N(p)$ and hence, $\Gamma(Z_{2p})$ is a star. Conversely, if G is a star then, $\gamma(\Gamma(Z_{2p})) = 1 = \gamma_c(\Gamma(Z_{2p}))$. ■

Theorem 4.5. For any graph $\Gamma(Z_n)$, $\gamma(\Gamma(Z_n)) = \gamma_c(\Gamma(Z_n))$ if and only if, $\Gamma(Z_n)$ has a spaning tree T with maximum number of pendent vertices such that for every set A of pendant vertices with $< A >$ independent of G, there exists a non-pendant vertex v in T such that $A \subseteq N(v)$.

Proof. If $\Gamma(Z_n)$ is a tree, using Theorem (4.2), the theorem is true. Let us consider $\Gamma(Z_n)$ is a connected graph with atleast one cycle. Then $\Gamma(Z_n)$ has a spanning tree T with a set A of pendant vertices such that D=V(T)-A. Since, $\gamma(\Gamma(Z_n)) = \gamma_c(\Gamma(Z_n))$ implies that, $n - |A|=|V(T) - A|$, where $|V(\Gamma(Z_n))| = n$. That is $\Gamma(Z_n)$ has a spanning tree T with maximum number of pendant vertices such that for every set A of pendant vertices with $< A >$ independent in $\Gamma(Z_n)$, then there exists a non pendant vertex v in T such that $A \subseteq N(v)$.

Conversely, if $\Gamma(Z_n)$ has a spanning tree T with maximum number of end vertices such that for every set A of pendant vertices with $< A >$ independnet in $\Gamma(Z_n)$, then there exists a non end vertex v in T such that $A \subseteq N(v)$. Clearly, D=V(T)-A. Hence, $\gamma(\Gamma(Z_n)) \leq n - |A| = \gamma_c(\Gamma(Z_n))$ implies that $\gamma(\Gamma(Z_n)) = \gamma_c(\Gamma(Z_n))$. ■

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