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On the Fekete-Szego Problem for a Subclass of λ-Convex Functions

Li Xiaofei

College of Technology and Engineering,
Yangtze University, Jingzhou-434020, China

Abstract

The purpose of the present paper is to introduce the classes $T^{\lambda}(\alpha)$ of normalized analytic and univalent functions in the open unit disc $U := \{z : |z| < 1\}$.

By using the properties of analytic functions and the technique of inequality in discussion,the paper is to derive the Fekete-Szego problem of the class $T^{\lambda}(\alpha)$.

Keywords: Analytic and Univalent; λ-convex functions; Fekete-Szego problem

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Introduction and Definition

Let Ω denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic and univalent in the open unit disc $U := \{z : |z| < 1\}$(for details, see [1,2,3]).Let $M(\lambda)$ denote λ-convex functions in U defined as follows (see [4]):

$$M(\lambda) = \left\{ f(z) \in \Omega : \mathrm{Re}\left((1-\lambda)\frac{zf^{/}(z)}{f(z)} + \lambda(1 + \frac{zf^{//}(z)}{f^{/}(z)}) \right) > 0, \lambda \geq 0 \right\}.$$

A classical theorem of Fekete and Szego[4] states that for $f(z) \in \Omega$ given by (1.1),

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} 3-4\mu, \mu \le 0 \\ 1+2e^{-\frac{2\mu}{1-\mu}}, 0 \le \mu \le 1 \\ 4\mu-3, \mu \ge 1 \end{cases}$$

This inequality is sharp in the sense that for each μ there exists a function in Ω such that equality holds. Pfluger [5,6] has considered the problem when μ is complex. In the case of C, S^*,and K, the subclasses of convex, starlike and close-to-convex functions,respectively,the above inequality can be improved [7,8].

In this paper,we define a subclass of λ-convex functions in U and reseach the Fekete-Szego problem of the class.

Definition 1.1: A function $f(z) \in \Omega$ given by (1.1) is said to be in the class $T^{\lambda}(\alpha)$ if the following condition is satisfied：

$$\left|\arg\left((1-\lambda)\frac{zf'(z)}{f(z)}+\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)\right)\right| < \frac{\alpha\pi}{2} \quad (0<\alpha \le 1, \lambda \ge 0, z \in U) \tag{1.2}$$

Main Results

To prove our main results, we need the following Lemma.

Lemma 2.1: [9] Let $p(z)$ be analytic in U and satisfy $\operatorname{Re}\{p(z)\} > 0$ for $z \in U$, with $p(z) = 1 + p_1 z + p_2 z^2 +$. Then

$$|p_n| \le 2 \quad (n \ge 1) \tag{2.1}$$

and

$$\left|p_2 - \frac{p_1^2}{2}\right| \le 2 - \frac{|p_1|^2}{2}. \tag{2.2}$$

The inequality (2.1) was first proved by Caratheodory [9](also,see Duren [1, p. 41]) and the inequality (2.2) can be found in [10, p.166].

With the help of Lemma 2.1, we now derive

Theorem 2.1: Let $f(z) \in T^{\lambda}(\alpha)$ and be given by (11). Then for complex number μ,

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} \dfrac{\alpha}{1+2\lambda}, k(\lambda) \le \dfrac{(1+\lambda)^2}{\alpha} \\ \dfrac{\left|\lambda^2+8\lambda+3-4\mu(1+2\lambda)\right|\alpha^2}{(1+2\lambda)(1+\lambda)^2}, k(\lambda) \ge \dfrac{(1+\lambda)^2}{\alpha} \end{cases}$$

where

$$k(\lambda)=\left|\lambda^2+8\lambda+3-4\mu(1+2\lambda)\right|.$$

For each μ, there is a function in $T^{\lambda}(\alpha)$ such that equality holds.

Proof: From (1.2), we can write the argument inequalities equivalently as follow:

$$(1-\lambda)\frac{zf'(z)}{f(z)}+\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)=[p(z)]^{\alpha}$$

where $p(z)$ is given by Lemma 2.1. Equating coefficents, we obtain

$$a_2=\frac{\alpha p_1}{1+\lambda}$$

and

$$a_3=\frac{1}{4(1+2\lambda)}\left(\alpha(\alpha-1)p_1^2+2\alpha p_2+\frac{2(1+3\lambda)\alpha^2p_1^2}{(1+\lambda)^2}\right) \tag{2.3}$$

Then we have

$$\left|a_3-\mu a_2^2\right|\le\frac{\alpha}{2(1+2\lambda)}\left(p_2-\frac{p_1^2}{2}\right)+\frac{\left(\lambda^2+8\lambda+3-4\mu(1+2\lambda)\right)\alpha^2p_1^2}{4(1+2\lambda)(1+\lambda)^2} \tag{2.4}$$

Hence (2.4) and Lemma 2.1 give

$$\left|a_3-\mu a_2^2\right|\le\frac{\alpha}{2(1+2\lambda)}\left(2-\frac{|p_1|^2}{2}\right)+\frac{\left|\lambda^2+8\lambda+3-4\mu(1+2\lambda)\right|}{4(1+2\lambda)(1+\lambda)^2}\alpha^2|p_1|^2$$

$$\le\frac{\alpha}{1+2\lambda}+\frac{\left\{\left|\lambda^2+8\lambda+3-4\mu(1+2\lambda)\right|\alpha^2-(1+\lambda)^2\alpha\right\}|p_1|^2}{4(1+2\lambda)(1+\lambda)^2}$$

Therefore, by using $|p_1|\le 2$, we have

$$\left|a_3-\mu a_2^2\right|\le\begin{cases}\dfrac{\alpha}{1+2\lambda},k(\lambda)\le\dfrac{(1+\lambda)^2}{\alpha}\\[2ex]\dfrac{\left|\lambda^2+8\lambda+3-4\mu(1+2\lambda)\right|\alpha^2}{(1+2\lambda)(1+\lambda)^2},k(\lambda)\ge\dfrac{(1+\lambda)^2}{\alpha}\end{cases}$$

where

$$k(\lambda)=\left|\lambda^2+8\lambda+3-4\mu(1+2\lambda)\right|.$$

Equality is attained for functions in $T^{\lambda}(\alpha)$, respectively, given by

$$(1-\lambda)\frac{zf'(z)}{f(z)}+\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)=\left(\frac{1+z^2}{1-z^2}\right)^{\alpha} \tag{2.5}$$

and

$$(1-\lambda)\frac{zf'(z)}{f(z)}+\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)=\left(\frac{1+z}{1-z}\right)^{\alpha} \tag{2.6}$$

Remark 2.1: It follows at once from (2.3) that $|a_2|\le 2\alpha/(1+\lambda)$.

Next, we consider the real number μ as follows.

Theorem 2.2: Let $f(z)\in T^{\lambda}(\alpha)$ and be given by (11). Then for real number μ,

$$|a_3-\mu a_2^2|\le\begin{cases}\dfrac{\alpha^2\left(\lambda^2+8\lambda+3-4\mu(1+2\lambda)\right)-(1+\lambda)^2}{(1+2\lambda)(1+\lambda)^2}, & \mu\le\dfrac{\alpha\left(\lambda^2+8\lambda+3\right)-(1+\lambda)^2}{4\alpha(1+2\lambda)}\\[2ex] \dfrac{\alpha}{1+2\lambda}, & \dfrac{\alpha\left(\lambda^2+8\lambda+3\right)-(1+\lambda)^2}{4\alpha(1+2\lambda)}\le\mu\le\dfrac{\alpha(\lambda^2+8\lambda+3)+(1+\lambda)^2}{4\alpha(1+2\lambda)}\\[2ex] \dfrac{\alpha^2\left(4\mu(1+2\lambda)-(\lambda^2+8\lambda+3)\right)}{(1+2\lambda)(1+\lambda)^2}, & \mu\ge\dfrac{\alpha\left(\lambda^2+8\lambda+3\right)+(1+\lambda)^2}{4\alpha(1+2\lambda)}\end{cases}$$

Proof: We consider two cases. At first, we suppose that $\mu\le\left(\lambda^2+8\lambda+3\right)/4(1+2\lambda)$.

Then (2.3) and Lemma 2.1 give

$$|a_3-\mu a_2^2|\le\frac{\alpha}{2(1+2\lambda)}\left(2-\frac{|p_1|^2}{2}\right)+\frac{\left(\lambda^2+8\lambda+3-4\mu(1+2\lambda)\right)}{4(1+2\lambda)(1+\lambda)^2}\alpha^2|p_1|^2$$

$$\le\frac{\alpha}{1+2\lambda}+\frac{\left(\lambda^2+8\lambda+3-4\mu(1+2\lambda)\right)\alpha^2-(1+\lambda)^2\alpha}{4(1+2\lambda)(1+\lambda)^2}|p_1|^2$$

So, by using the fact that $|p_1|\le 2$, we obtain

$$|a_3-\mu a_2^2|\le\begin{cases}\dfrac{\alpha}{1+2\lambda}, & \dfrac{\alpha\left(\lambda^2+8\lambda+3\right)-(1+\lambda)^2}{4\alpha(1+2\lambda)}\le\mu\le\dfrac{\lambda^2+8\lambda+3}{4(1+2\lambda)}\\[2ex] \dfrac{\alpha^2\left(\lambda^2+8\lambda+3-4\mu(1+2\lambda)\right)-(1+\lambda)^2}{(1+2\lambda)(1+\lambda)^2}, & \mu\le\dfrac{\alpha\left(\lambda^2+8\lambda+3\right)-(1+\lambda)^2}{4\alpha(1+2\lambda)}\end{cases}$$

Equality is attained by choosing $p_1=p_2=2$ and $p_1=0, p_2=2$, respectively, in (2.3).

Next, we suppose that $\mu\ge\left(\lambda^2+8\lambda+3\right)/4(1+2\lambda)$. In this case, it follows again from (2.3) and Lemma 2.1 that

$$\left|a_3-\mu a_2^2\right| \leq \frac{\alpha}{2(1+2\lambda)}\left(2-\frac{|p_1|^2}{2}\right)+\frac{\left(4\mu(1+2\lambda)-(\lambda^2+8\lambda+3)\right)}{4(1+2\lambda)(1+\lambda)^2}\alpha^2|p_1|^2$$

$$\leq \frac{\alpha}{1+2\lambda}+\frac{\left(4\mu(1+2\lambda)-(\lambda^2+8\lambda+3)\right)\alpha^2-\alpha(1+\lambda)^2}{4(1+2\lambda)(1+\lambda)^2}|p_1|^2$$

and so, as in the first case, we have

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \dfrac{\alpha}{1+2\lambda}, \dfrac{\lambda^2+8\lambda+3}{4(1+2\lambda)} \leq \mu \leq \dfrac{\alpha(\lambda^2+8\lambda+3)+(1+\lambda)^2}{4\alpha(1+2\lambda)} \\ \dfrac{\alpha^2\left(4\mu(1+2\lambda)-(\lambda^2+8\lambda+3)\right)}{(1+2\lambda)(1+\lambda)^2}, \mu \geq \dfrac{\alpha\left(\lambda^2+8\lambda+3\right)+\left(1+\lambda\right)^2}{4\alpha(1+2\lambda)} \end{cases}$$

The results are sharp by choosing $p_1=0, p_2=2$ and $p_1=2i, p_2=-2$, respectively, in (2.3).

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