International Journal of Mathematics Research.
ISSN 0976-5840 Volume 5, Number 1 (2013), pp. 41-48

http://www.irphouse.com

Comparative Study of Double Mellin-Laplace and Fourier Transform

Sarita poonia[1*] and Dr Rachna Mathur[2]

*[*1]Research Scholar, Mathematics Department Dungar Collage, Bikaner(Raj.)*
[2]Hed of Mathematics Department Dungar Collage, Bikaner (Raj.)

Abstract:

The aim of this paper is to drive the relation between the Finite Mellin Integral Transform with the Laplace Transform by using the double Laplace and Fourier Mellin integral Transform. Properties like Linearity Property, Scaling Property, Power Property and f(ax)g(by) are also derived. The Shifting and Inversion Theorem for Laplace –Finite Mellin Integral Transform and Shifting Property discussed for Fourie –Mellin Integral Transforms.

Mathematics Subject Classification: 44A10, 47D03, 46O50, 47G30, 33C45, 44A35,
26D99

Keywords: Laplace Transform, Fourier Transform, Mellin Integral Transform, Finite Mellin Integral Transforms.

1:Introduction

The Laplace Transform and Fourier Transform are widely used for solving differential and Integral equations. In physics and engineering it is used for analysis of linear time invariant systems such as electrical circuits, harmonic oscillators, optical devices and mechanical systems. In this analysis, the Laplace Transform is often interpreted as transformation from the time domain in which input and output functions of time to the frequency domain, where the same inputs and outputs are functions of complex annular frequency in radians per unit time. Given a simple mathematical or functional description of an input or output to a system. The Laplace transform provides an alternative functional as description that often simplifies the process of analyzing the behaviour of the system or in synthesizing a view system based on a set of specifications. Fourier Transform is often use in signal processing.

The theory of integral has presented a direct and systematic technique for the resolution of certain type of classical boundary and intial value problems. To be successful the transform must be adopted to the form of the differential operator to be eliminated as well as to the range of interest and the associated boundary conditions. There are numerous cases for which no suitable transform exists.

Here we consider Laplace Finite Mellin integral Transform to the removal of the polar operators that occur when Laplace operator is expressed in either spherical or plan polar coordinates.

The Double Laplace Transform can be used to find the Laplace –Finate Mellin Integral Transform in the range [0, 1] and [0, 1]. Fourier Laplace Transform is used to find the Fourier Mellin Integral Transform in the rang $[0, \infty)$ and $[0, \infty)$.

2: Preliminary Results

2. 1: Relation Of The Finite Mellin Integral Transform With Laplace Transform

The Laplace Integral Transform of the function $f(x)$ of x is denoted by $L[f(x), s]$ and defined as

$$F(s) = L[f(x), s] = \int_0^\infty e^{-sx} f(x)dx$$

Whenever this integral is exists for $s > 0$ is the parameter.

The inverse of the Laplace Transform is denoted by $L^{-1}[F(s), x] = f(x)$ and defined as

$$f(x) = L^{-1}[F(s), x] = \frac{1}{2\pi i} \int_{c_1 - i\infty}^{c_1 + i\infty} e^{sx} L[f(x), s]ds$$

The Mellin Integral Transform of the function $f(y)$ of y is denoted by $M[f(y), r]$ and defined as

$$M(r) = M[f(y), r] = \int_0^\infty y^{r-1} f(y)dy$$

Whenever this integral is exists for $r > 0$ is the parameter.

The inverse of the Mellin Transform is denoted by $M^{-1}[F(r), y] = f(y)$ and defined as

$$f(y) = M^{-1}[F(r), y] = \frac{1}{2\pi i} \int_{c_2 - i\infty}^{c_2 + i\infty} y^{-r} M[f(y), r]dr$$

The Finite Mellin Integral Transform of the function $f(y)$ of y is denoted by $M[f(y), r]$ and defined as

$$M(r) = M[f(y), r] = \int_{0}^{1} y^{r-1} f(y) dy$$

Whenever this integral is exists for $r > 0$ is the parameter.

Its inverse is defined as

$$f(y) = M_f^{-1}[F(r), y] = \frac{1}{2\pi i} \int_{c_2-i}^{c_2+i} y^{-r} M_f[f(y), r] dr$$

The Double Laplace Transform is denoted by $L_2[f(x,z), s, r]$ and defined as

$$L_2[f(x,z), s, 0, \infty, r, 0, \infty] = \int_{0}^{\infty} \int_{0}^{\infty} e^{-(sx+rz)} f(x,z) dx dz$$

Whenever this Double Integral is exists for $r > 0$ and $s > 0$ are parameters. Substitute $\log y = -z$ and $\log t = -x$ then Double Laplace Integral Transform is

$$L_2[f(t,y), s, 0, 1, r, 0, 1] = \int_{0}^{1} \int_{0}^{1} t^{s-1} y^{r-1} f(t,y) dt dy$$

This is relation between Finite Mellin Integral and Laplace Transform for $f(t,y)$ with parameters

$r > 0$ and $s > 0$ in the range [0, 1] and [0, 1] and is denoted as

$${}_{2f}^{\,l}M[f(t,y), s, 0, 1, r, 0, 1] = \int_0^1 \int_0^1 t^{s-1} y^{r-1} f(t,y) dt dy \tag{1}$$

Where $0 \le t \le 1$ and $0 \le y \le 1$

2. 2: Relation Of The Mellin Integral Transform With Fourier Transform

The Fourier Transform is denoted by $F[f(x), s] = F(s)$ and can be defined as

$$F[f(x), s] = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x) dx$$

Whenever this integral is exists for $s > 0$ is the parameter.

The inverse of the Fourier Transform is denoted by $F^{-1}[F(s), x] = f(x)$ and defined as

$$F^{-1}[F(s), x] = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isx} F(s) ds$$

Whenever parameter $s > 0$.
The Mellin Integral Transform in the following way

$$M[f(x), r] = \int_0^\infty x^{r-1} f(x)dx, \alpha < Re(s) < \beta$$

Where α and β are real numbers deterring the maximum range of values of $Re(z)$ such that the integral converges. The Fourier Laplace Transform $FL[f(x,z), s, r]$ can be defined as

$$FL[f(x,z), s, -\infty, \infty, r, 0, \infty] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_0^{\infty} e^{isx-rz} f(x,z)dxdz$$

Substitute $\log y = -z$ and $\log t = ix$ then

$$FL[f(t,y), s, 0, \infty, r, 0, 1] = \frac{-i}{\sqrt{2\pi}} \int_0^\infty \int_0^1 t^{s-1} y^{r-1} f(t,y)dtdy$$

$$FF[f(x,z), s, -\infty, \infty, r, -\infty, \infty] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{isx} e^{irz} f(x,z)dxdz$$

$$= \frac{-1}{2\pi} \int_0^\infty \int_0^\infty t^{s-1} y^{r-1} f(t,y)dtdy$$

$$= \frac{-1}{2\pi} {}_2^f M[f(t,y), s, 0, \infty, r, 0, \infty] \tag{2}$$

This is Fourier Mellin Integral Transform in range $[0, \infty)$ and $[0, \infty)$ and $0 \le x < \infty$ and $0 \le z < \infty$

3:LEMMAS(Laplace –Finite Mellin Integral Transform-LFMIT)

3. 1:Lemma. 1

3. 1. 1:The LFMIT is

$${}_{2f}^{l}M[f(t,y), s, 0, 1, r, 0, 1] = \int_0^1 \int_0^1 t^{s-1} y^{r-1} f(t,y)dtdy$$

Then Linearity property

$${}_{2f}^{l}M[\alpha f(x,y) \pm \beta g(x,y), s, 0, 1, r, 0, 1] = \alpha\, {}_{2f}^{l}M[f(x,y), s, 0, 1, r, 0, 1] \pm \beta\, {}_{2f}^{l} M[g(x,y), s, 0, 1, r, 0, 1]$$

3. 1. 2: Scaling property

$${}_{2f}^{l}M[f(x, y^n), s, 0, 1, r, 0, 1] = \frac{1}{n}\, {}_{2f}^{l}M[f(x,z), s, 0, 1, \frac{r}{n}, 0, 1] \tag{4}$$

3. 2:Lemma. 2

3. 2. 1:

$$ {}_{2f}^{l}M[f(cx,dy),s,0,1,r,0,1] = \frac{1}{c^s d^r}\int_0^c\int_0^d z^{s-1}\,t^{r-1}f(z,t)dzdt $$

$$ =\frac{1}{c^s d^r}\,{}_{2f}^{l}M[f(z,t),s,0,c,r,,0,d] \quad (5) $$

3. 2. 2:

$$ {}_{2f}^{l}M[f(cx)g(dy),s,0,1,r,0,1] = \frac{1}{c^s d^r}M[f(t),s,0,c]M[g(z),r,0,d] \quad (6) $$

4:LEMMAS(Fourier Mellin Integral Transform-FMIT)

4. 1:Lemma

Let $f(x,y), x,y \in R_+$,

$$ FF[f(x,z),s,-\infty,\infty,r,-\infty,\infty] = \frac{-1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{isx}\,e^{irz}f(x,z)dxdz $$

$$ FF[f(x,z),s,-\infty,\infty,r,-\infty,\infty] = \frac{-1}{2\pi}\,{}_{2}^{f}M[f(t,y),s,0,\infty,r,0,\infty] $$

$$ {}_{2}^{f}M[\alpha f(x,y) \pm \beta g(x,y),s,0,\infty,r,0,\infty] = \alpha\,{}_{2}^{f}M[f(x,y),s,0,\infty,r,0,\infty] \pm \beta\,{}_{2}^{f}M[g(x,y),s,0,\infty,r,0,\infty] \quad (7) $$

4. 2:Lemma

$$ {}_{2}^{f}M[f(x,y^n),s,0,\infty,r,0,\infty] = \frac{1}{n}\,{}_{2}^{f}M[f(x,z),s,0,\infty,\frac{r}{n},0,\infty] \quad (8) $$

4. 3:Lemma

$$ {}_{2}^{f}M[f(cx,dy),s,0,\infty,r,0,\infty] = \frac{1}{c^s d^r}\,{}_{2}^{f}M[f(z,t),s,0,\infty,r,0,\infty] \quad (9) $$

4. 4:Lemma

$$ {}_{2}^{f}M[f(cx)g(dy),s,0,\infty,r,0,\infty] = \frac{1}{c^s d^r}M[f(t),s,0,\infty]M[g(z),r,0,\infty] \quad (10) $$

5:SHIFTING THEOREMS

5. 1 Shifting Theorem For Laplace-Finite-Mellin Integral Transform

The Shifting Theorem For Laplace-Finite Mellin Integral Transform is

${}_{2f}^{l}M[f(t,y),s,0,1,r,0,1] = \int_0^1\int_0^1 t^{s-1}\,y^{r-1}f(t,y)dtdy$ then

$$ {}_{2f}^{l}M[x^m y^n f(x,y),s,0,1,r,0,1] = $$

$$ {}_{2f}^{l}M[f(x,y),m+s,0,1,n+r,0,1] \quad (11) $$

5. 2 Shifting Theorem For Fourier-Mellin Integral Transform

The Shifting Theorem For Fourier Mellin Integral Transform is

$$FF[f(x,z),s,-\infty,\infty,r,-\infty,\infty] = \frac{-1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{isx}\, e^{irz} f(x,z)dxdz$$

$$FF[f(x,z),s,-\infty,\infty,r,-\infty,\infty] = \frac{-1}{2\pi}\,{}_2^fM[f(t,y),s,0,\infty,r,0,\infty]$$

$${}_2^fM[\,x^m y^n f(x,y),s,0,\infty,r,0,\infty] = \frac{-1}{2\pi}$$

$${}_2^fM[f(x,y),m+s,0,\infty,n+r,0,\infty] \qquad (12)$$

6:Inversion Theorem

6. 1:Inversion Theorem For The Laplace-Finite Mellin Integral Transform

The Laplace Integral Transform of the function $f(x)$ of x is denoted by $L[f(x),s]$ and defined as

$$F(s) = L[f(x),s] = \int_0^{\infty} e^{-sx} f(x)dx$$

Whenever this integral is exists for $s > 0$ is the parameter.

The inverse of the Laplace Transform is denoted by $L^{-1}[F(s),x] = f(x)$ and defined as

$$f(x) = L^{-1}[F(s),x] = \frac{1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty} e^{sx}\, L[f(x),s]ds$$

The Mellin Integral Transform of the function $f(y)$ of y is denoted by $M[f(y),r]$ and defined as

$$M(r) = M[f(y),r] = \int_0^{\infty} y^{r-1} f(y)dy$$

Whenever this integral is exists for $r > 0$ is the parameter.

The inverse of the Mellin Transform is denoted by $M^{-1}[F(r),y] = f(y)$ and defined as

$$f(y) = M^{-1}[F(r),y] = \frac{1}{2\pi i}\int_{c_2-i\infty}^{c_2+i\infty} y^{-r}\, M[f(y),r]dr$$

For the Laplace-Finite Mellin transform is

$${}_{2f}^{\,l}M[f(t,y),s,0,1,r,0,1] = \int_0^1\int_0^1 t^{s-1}\, y^{r-1} f(t,y)dtdy$$

And its inversion formula is

$$f(x,y) = \frac{1}{(2\pi i)^2} \int_{c_2-i}^{c_2+i} \int_{c_1-i}^{c_1+i} \frac{x^{-s}y^{-r}}{sr} \, {}_{2f}^{\,l}M[f(x,y),s,0,1,r,0,1]drds$$

Proof

Assume that ${}_{2f}^{\,l}M[f(x,y),s,0,1,r,0,1]$ is a regular function in the strips $|Re(s)| < r$ ($'r'$ to be real number) of the s-plane and that $0 < c < v_1, c_1 - i \le s \le c_1 + i$ where $c_1$is constants and $|Re(p)| < q$ ($'q'$ to be real number) of the p-plane and that $0 < c < v_2, c_2 - i \le p \le c_2 + i$ where c_2 is constants,

If the Laplace-Finite Mellin integral transform is

$${}_{2f}^{\,l}M[f(x,y),s,0,1,r,0,1] = \int_0^1 \int_0^1 x^{s-1}\, y^{r-1} f(x,y)dxdy$$

$$= \int_0^1 \int_0^1 x^{s-1}\, y^{r-1} [\frac{1}{(2\pi i)^2} \int_{c_2-i}^{c_2+i} \int_{c_1-i}^{c_1+i} x^{-s}y^{-r}\, {}_{2f}^{\,l}M[f(x,y),s,0,1,r,0,1]drds]dxdy$$

$$= \frac{1}{(2\pi i)^2} [\int_{c_2-i}^{c_2+i} \int_{c_1-i}^{c_1+i} x^{-s}y^{-r}\, {}_{2f}^{\,l}M[f(x,y),s,0,1,r,0,1]drds] \left[\frac{x^s}{s}\right]_0^1 \left[\frac{y^r}{r}\right]_0^1$$

$$= \frac{1}{(2\pi i)^2} \int_{c_2-i}^{c_2+i} \int_{c_1-i}^{c_1+i} x^{-s}y^{-r}\, {}_{2f}^{\,l}M[f(x,y),s,0,1,r,0,1]drds] \frac{1}{sr}$$

$$f(x,y) = \frac{1}{(2\pi i)^2} \int_{c_2-i}^{c_2+i} \int_{c_1-i}^{c_1+i} \frac{x^{-s}y^{-r}}{sr} \, {}_{2f}^{\,l}M[f(x,y),s,0,1,r,0,1]drds] \quad (13)$$

References

[1] A. H. Zemanian The Distributional Laplace and Mellin Transformations, J. SIAM Vol. 14. No. 1. Jan. 1908 Prited in U. S. A.

[2] C. Fox Application of Mellin Transformation to Integral Equation, 3rd March, 1934, pp 495-502

[3] C. Fox, Applications of Mellin’s Transformation to the integral equations, (1933)

[4] Derek Naylor, On a Mellin Type Integral Transforms, Journal of Mathematics and Mechanics, (1963) vol. 12, No. 2

[5] D. J. Bedingham, Dimensional Regularization and Mellin summation in High-Temperature Calculations, Therotical Physics, The Blackett Laboratory, Imperial College, Prince Consort Road, SW7 2BW, UK. Ar Xiv:ph/00110112x1 1NOV 2000.

[6] Ian N. Sneddon, The use of Integral Transforms, TMH edition 1974

[7] J. M. Mendez and J. R. Negrin, On the finite Hankel –Swchwartz Transformation of Distributions, Ganta. Vol. 39, No. 1. 1908
[8] I. S. Reed, The Mellin Type Double Integral, Cambridge, London.
[9] Lokenath Debnath, Integral Transforms And Their Applications, 1995, CRCPress Inc.
[10] S. Derrode, ”Robust and Efficient Fourier-Mellin Transform Appoximations for Gray – Level image Reconstruction and Complete invariant Desciption ” Computer Vision and Image U 83, 57-78(2001)
[11] S. M. Khairnar, R. M. Pise & J. N. Salunke, “Study of The Sumudu Mellin Integral Transform and its Applications”, Int. J. of Mat. Sci. & Engg. Appl. (IJMSEA) Vol. 4, No. IV, (October. 2010), pp-307-325
[12] S. M. Khairnar, R. M. Pise & J. N. Salunke, Relation of Finite Mellin Integral Transform with Laplace and Fourier Transforms, Contemporary Engineering Sciences, Vol. 4, 2011, no. 6, 269 – 288.