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On Characterization Non-Commutative Structure of Spin Manifold

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ABSTRACT

Our main aim is to derive the non-commutative structures of manifolds using spectral triples. The triples in the spin^c case, strengthen usual conditions given by A. Connes[2]. we derive to spin^c condition with a Riemannian condition. We show that such a Riemannian spectral triple represents a generalization of Kasparov's[4] fundamental class in KK-theory. We establish here relations Kasparov's translation between the spin^c and Riemannian Poincare duality isomorphisms.

Key–word: "(Spectral, Spin manifold, Poincare duality, Kaspaov product, Isomorphic)"

1.1 Introduction

We discuss here some conditions required for formulation of non-commutative structure.

1. Let A and B denote separable unital C*-algebras. Whereas $\mathcal{A}$ and $\mathcal{B}$ denote dense *-subalgebras $\mathcal{A} \subset$ A and $\mathcal{B} \subset$ B. $\mathcal{A}$ is equipped with a locally convex topology finer than that given by the C*-norm of A, and similarly for $\mathcal{B}$ and B.
2. In a C*-algebra A, a $\geq$ 0 means that α is a positive element of the C*-algebra A, let us write a > 0 when a is a nonzero positive element of A.
3. For any algebra $\mathcal{A}$, its opposite algebra is denoted by $\mathcal{A}^o$ with elements a^o, b^o, satisfying the condition $a^o b^o = (ba)^o$.
4. Let us choose "inner products": in two different ways (i) hermitian pairings with values in a *-algebra such as $(e|\ f)_{\mathcal{A}}$ or $_{\mathcal{B}}(e\ |\ f)$, and (ii) scalar products of vectors in a Hilbert space have angle brackets, like $\langle \xi \mid \eta \rangle$.

5. Zhang and Cacie introduced “almost commutative" spectral triples. The spectral triple over $C^\infty(M)$ in the non-spin^c case, by taking twisted K-theory to analyse orientability. Boeijink and van Suijlekom formulated a spectral triple over the Clifford algebra bundle.
6. Let $(\mathcal{A},\mathcal{H},\mathcal{D})$ denote a spectral triple over a trivially graded unital *-algebra $\mathcal{A}$, topologised as a separable Frechet algebra for which the operator norm on $\mathcal{B}(\mathcal{H})$ is continuous. The following conditions introduced by Connes enable us to derive a reconstruction theorem for compact spin^c manifolds.
   a. (Regularity). The spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is QC^∞, $\mathcal{A}$ is then a Frechet pre-C^*-algebra.
   b. (Dimension). The spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is Z_p-summable for a fixed positive integer p. Thus, if Tr_ω denotes any Dixmier trace corresponding to a Dixmier limit ω, the linear functional $\psi_\omega(a) := Tr_\omega(a\ \langle\mathcal{D}\rangle^{-p})$ is defined (and positive) on A.

1.2 Theorem

Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spectral triple satisfying (i) and (ii). Let us further that $\mathcal{H}_\infty$ is finite projective over $\mathcal{A}$, so that $\mathcal{H}_\infty \approx p\mathcal{A}^n$ for some projector $p \in M_n(\mathcal{A})$. Also, $\mathcal{A}$ commutes with p and that only scalars commute with all $a \in A$ and $\mathcal{D}$. Then any right Hermitian pairing on $\mathcal{H}_\infty \approx p\mathcal{A}^n$ satisfying conditions (i)and (ii) is a positive multiple.

Proof

By an appropriate application assertion of theorem (1.2) there is a positive invertible element $r \in pM_n(\mathcal{A})p$ such that the given pairing is of the form, i.e., $(\xi|\eta)_r \equiv \sum_{j,k} a_j^* r_{jk} b_k$ for $\xi = \sum_j \xi\ a_j$, $\eta = \sum_k \xi_k\ b_k$ in $\mathcal{H}_\infty$. If $a \in \mathcal{A}$, then since $ap = pa$ we get $a^*\xi = \sum_j \xi_j a^* a_j$ and $a\eta = \sum_k \xi_k\, ab_k \in \mathcal{H}_\infty$. The formula reduces it to the form.

$$\langle\xi|a\eta\rangle = \langle\alpha{*}\eta|\eta\rangle = Tr_\omega((a^*\xi|\eta)_r\langle\mathcal{D}\rangle^{-p} = Tr_\omega(\xi|r^{-1}\ ar\eta)_r\ \langle\mathcal{D}\rangle^{-p} = \langle\xi\ |r^{-1}ar\eta\rangle \ldots (1)$$

Hence $[r, a] = 0$ for all $a \in \mathcal{A}$.

Now since $\mathcal{D}$ is a selfadjoint operator on $\mathcal{H}$, we obtain, for $\xi,\eta\ 2{\in}H_\infty$:

$$0 = \langle\mathcal{D}\xi|\eta\rangle - \langle\xi|\mathcal{D}\eta\rangle = Tr_\omega\ (((\mathcal{D}\xi\ |\eta)_r - (\xi|\mathcal{D}\eta)_r)\ \langle\mathcal{D}\rangle^{-p})$$

$$= Tr_\omega\ (((\mathcal{D}\xi\ |r\eta) - (\xi|r\mathcal{D}\eta))\ \langle\mathcal{D}\rangle^{-p}) = :\ \langle\langle\mathcal{D}\xi\ |r\eta\rangle\rangle - \langle\langle r\xi\ |\mathcal{D}\eta\rangle\rangle \ldots \qquad (2)$$

where $\langle\langle\xi|\eta\rangle\rangle := \langle r^{-1}\xi|\eta$ defines a new Hilbert space scalar product. Since r^{-1} is bounded with bounded inverse, this scalar product $\langle\langle.|.\rangle\rangle$ is topologically equivalent to the old one $\langle.|.\rangle$, and so $\mathcal{H}$ coincides with the completion of H_∞ with respect to either scalar product. Now the right hand side$^{(2)}$ is the quadratic form defining the commutator $[\mathcal{D}, r]$ with respect to the scalar product $\langle\langle.|.\rangle\rangle$. It vanishes on H_∞ and thus $[\mathcal{D}, r] = 0$. The irreducibility condition now implies that r is (a positive multiple of) the identity p in $pM_n(\mathcal{A})p$, represented by a scalar operator on $\mathcal{H}$. Hence, the theorem is proved.

1.3 Theorem

Let $(\mathcal{A},\mathcal{H},\mathcal{D},c)$ be a noncommutative oriented spin^c manifold such that only scalars commute with all $a \in A$ and $\mathcal{D}$. Then H_∞ is finite projective as both a left and a right $\mathcal{A}$-module, and $(\mathcal{A})$ is finite projective as a left or right $\mathcal{A}$-module. equavalently, the relations

$$\langle\eta|\xi\rangle = \psi_\omega((\eta|\eta\xi)_{\mathcal{A}}) = Tr_\omega((\eta|\xi)_{\mathcal{A}}\langle\mathcal{D}\rangle^{-p}$$

hold for all ξ, $\eta \in H_\infty$. In particular, ψ_ω is faithful on $\mathcal{A}^O$.

Proof

Let us put $H_\infty = \mathcal{C}^m q$, combining the condition (ii) with theorem (1.2) we get $H_\infty \simeq p\mathcal{A}^n$ and $\mathcal{C} = pM_n(\mathcal{A})p$. Since $\mathcal{A} \subset \mathcal{C} = \mathcal{C}(\mathcal{A})$, $\mathcal{A}$ commutes with the projector p. That property ensures that the partial trace tr : $\mathcal{C} \to \mathcal{A}$, defined on $T = [t_{ij}] \in pMn(\mathcal{A})p$ by $tr(T) := \sum_{i=1}^{n} t_{ii}$, is a well-defined operator-valued weight. It also shows that $\mathcal{C}$ is finite projective as a left or right $\mathcal{A}$-module, because $pM_n(\mathcal{A})p \subset M_n(\mathcal{A})$ is a direct summand as an A-module precisely because p commutes with the action of $\mathcal{A}$. We thud find that H_∞ is a finite projective left module over A, since $H_\infty = \mathcal{C}^m q$ is a direct summand in $\mathcal{C}^m$, is also a direct summand in $\mathcal{A}^{mn2}$. But taking help of the assertion of theorem (1.3) the left $\mathcal{A}$-valued inner product on H_∞ is given on $\xi = \Sigma_j \xi_j a_j$ and $\eta = \Sigma_k \eta_k b_k$ which may expressed as

$$(\xi|\eta)\mathcal{A} = \lambda \sum_{j,k} a_i^* \; a_{ij} b_i, \qquad \text{for some } \lambda > 0$$

$$\Rightarrow Tr_\omega((\xi|\eta)_{\mathcal{A}}\langle\mathcal{D}\rangle^{-p}) = \lambda \sum_{i,j} Tr_\omega(a_i^* \; a_{ij} b_i \langle\mathcal{D}\rangle^{-p})$$

$$= \lambda \sum_{i,j} Tr_\omega(p_{ij} \; b_j a_k^* \; p_{kj} b_i \langle\mathcal{D}\rangle^{-p})$$

$$= Tr_\omega \; (tr(\mathcal{C} \; (\eta|\xi)\langle\mathcal{D}\rangle^{-p}) \qquad \ldots(3)$$

the $\mathcal{C}_{\mathcal{D}}(\mathcal{A})$ inner product is determined by the Morita equivalence condition. We conclude here that $\mathcal{H} \simeq pL^2(\mathcal{A}, \psi_\omega)^n$ and trace is precisely $Tr_{\mathcal{H}} = Tr_{L}{}^2(\mathcal{A},\psi_\omega)^o$ tr, in view of the positivity condition of trace to $\mathcal{C}_{\mathcal{D}}(\mathcal{A})$. Hence, the theorem is proved.

$$Tr_\omega((\xi\;|\eta)_{\mathcal{A}} \langle\mathcal{D}\rangle^{-p}) = Tr_\omega \; (\mathcal{C}\left(\eta|\xi)\langle\mathcal{D}\rangle^{-p}\right) = \langle\xi|\eta\rangle$$

1.4 Theorem

Let $(\mathcal{A},\mathcal{H},\mathcal{D},c)$ be a p-dimensional noncommutative spin^c manifold with

Kasparov class $\mu \in KK^p(A \otimes A^o, C) \simeq KK(A \otimes A^o \otimes C\ell^p{}_1, C)$. Regard the conjugate module $(H_\infty \otimes C_p{}^2)^{]}$ as an $(A \otimes C_1{}^p)$-$\mathcal{C}(\mathcal{A})$-bimodule, graded in odd spectral dimensions, with class $\sigma \in KK(C^o, \otimes A^o C^p{}_1)$. Then $\lambda := \sigma \otimes_{AoC^p{}_1} \mu \in KK(A \otimes C^o, C)$ is the class of a non-commutative Riemannian manifold. If μ satisfies spin^c Poincare duality, then λ satisfies Riemannian Poincare duality.

Proof

Let us first prove the even case the odd case follows after modifications. We begin with the noncommutative spin^c manifold $((\mathcal{A}, \mathcal{H}, \mathcal{D}, c)$ and the pre-Morita equivalence bimodule H_∞ between $\mathcal{C}_{\mathcal{D}}(\mathcal{A})$ and $\mathcal{A}$ given by the spin^c condition. Let us identify $H_\infty = q\mathcal{A}^m$. The conjugate module H_∞ gives a pre-Morita equivalence between $\mathcal{A}$ and $\mathcal{C}(\mathcal{A}) \simeq qM^m(\mathcal{A})q$. The spectral triple $(\mathcal{A}, \mathcal{H} \otimes_{\mathcal{A}} H^{\parallel}{}_1, \widehat{\mathcal{D}}) = (A, H^m q, \widehat{\mathcal{D}})$, which satisfies the first-order condition. It implies that

$$\langle\widehat{\mathcal{D}}\rangle^{-p} = q^o(\langle\widehat{\mathcal{D}}\rangle^{-p} \otimes 1_m)q^o + B, \text{ where } B \in \mathcal{L}_0^{1\infty}(\mathcal{H}^m q)$$

Let us now prove that B has vanishing Dixmier trace, Put $\mathcal{D}'_m = q^o(\mathcal{D} \otimes 1_m)q^o$, so that $\widehat{\mathcal{D}} = \mathcal{D}'_m + \hat{A}$ with $\hat{A}$ bounded. we find that $(i + \widehat{\mathcal{D}})^{-1}$ $(i + \mathcal{D}'_m)^{-1}$ mod $Z_{p/2}$, and thus $\langle\widehat{\mathcal{D}}\rangle^{-1} \equiv \langle\mathcal{D}'_m\rangle^{-1}$ mod $Z_{p/2}$.

The operator trace over $\mathcal{H}^m q$ is $Tr_{\mathcal{H}} \otimes tr_m(q^o(.)q^o)$ with tr_m denoting a matrix trace. The left action of $\mathcal{C}(\mathcal{A})$ commutes with q^o, thus, if w is an even element of $\mathcal{C}_{\mathcal{D}}(\mathcal{A})$ we get an equality of Dixmier traces:

$$\boldsymbol{Tr}_{\omega}^{\mathcal{H}m_q}(\omega\langle\widehat{\boldsymbol{\mathcal{D}}}\rangle^{-p}) = \boldsymbol{Tr}_{\omega}^{\mathcal{H}m_q}(\omega q^o\langle\widehat{\boldsymbol{\mathcal{D}}}\rangle^{-p}) = \boldsymbol{Tr}_{\omega}^{\mathcal{H}}(\omega tr^m(q^o)\langle\boldsymbol{\mathcal{D}}\rangle^{-p}) \quad \ldots(4)$$

We must now show that the new spectral triple satisfies Condition. The spin^c condition, namely that H_∞ is a pre-Morita equivalence bimodule between $\mathcal{C}(\mathcal{A})$ and A, shows that there are finitely many vectors ξ_j , $\eta_j \in \mathcal{H}_\infty$ such that

$$\ominus_{\xi_1\eta_1} + \ldots + \ominus_{\xi_m\eta_m} = \mathbf{1} \in \boldsymbol{\mathcal{C}_{\mathcal{D}}(\mathcal{A})} \quad \ldots(5)$$

Consider the vector $\Phi \in \mathcal{H} \otimes_A \mathcal{H}^{\parallel}{}_\infty$ defined by

$$\Phi := \xi_1 \otimes \eta_1^{\parallel} + \ldots + \xi_m \otimes \eta_m^{\parallel}$$

We prove that Φ is an algebraically cyclic vector for $\mathcal{C}(\mathcal{A})$, and that the vector state $\sigma_\Phi : \omega \rightarrow \langle\Phi \mid \omega\Phi\rangle$ is of the form

$$\sigma_\Phi(\omega) = \boldsymbol{Tr}_{\omega}^{\mathcal{H}m_q}(\omega z\langle\widehat{\boldsymbol{\mathcal{D}}}\rangle^{-p}) \quad \ldots(6)$$

for a central element $z \in \mathcal{C}_{\mathcal{D}}(\mathcal{A})$. Under the isomorphism $\Lambda : \ominus_{\xi\eta} \rightarrow \xi \otimes \eta^{\parallel} : \mathcal{C}_{\mathcal{D}}(\mathcal{A}) \rightarrow \mathcal{H}_\infty \otimes_A \mathcal{H}^{\parallel}{}_\infty$, the vector Φ is just the image $\Lambda(1)$ of the unit element of $\mathcal{C}_{\mathcal{D}}(\mathcal{A})$.

$$\mathbf{1} = \sum_{k} \ominus_{\xi_k \eta_k} = \mathbf{1}^2 \sum_{j\ k} \ominus_{\xi_k (n_k | \xi_j |)} \boldsymbol{a}, \qquad \eta_i = \sum_{j\ k} \ominus_{\xi_k (n_k | \xi_j |)} \boldsymbol{a}, \qquad \eta_i \quad \dots (7)$$

so that $\eta_k = \Sigma_j\ \eta_j(\xi_j \mid \eta_k)_A$ for each k. Moreover, if $w = \Sigma_{i,k}\ \ominus\xi_i a_i, \eta_k b_k \in \mathcal{C}_{\mathcal{D}}(\mathcal{A})$, then it reduces to the form

$$\omega\Phi = \sum_{i,j,k} \xi_i \boldsymbol{a}_i \left(\eta_k \boldsymbol{b}_k \ \middle|\xi_j\right)\ _{\mathcal{A}} \otimes\ \eta_j^{\|} = \sum_{i,j,k} \xi_i \boldsymbol{a}_i\, \boldsymbol{b}_k^* \left(\eta_k \ \middle|\xi_j\right)\ _{\mathcal{A}} \otimes\ \eta_j^{\|}$$

$$= \sum_{i,j,k} \xi_i \boldsymbol{a}_i\, \boldsymbol{b}_k^* \otimes\ \eta_j^{\|} \ = \sum_{i,k} \xi_i \boldsymbol{a}_i \otimes (\eta_k \boldsymbol{b}_k)^{\|} = \Lambda(\omega) \quad \dots (8)$$

Thus $w \to w\Phi = \Lambda(w)$ maps $\mathcal{C}(\mathcal{A})$ onto $H_\infty \otimes_A \mathcal{H}^{\|}_\infty$, so that Φ is algebraically cyclic, and it is separating as well. since the pre-Morita equivalence implies that Λ is bijective. The scalar product on the Hilbert space $\mathcal{H} \otimes_A \mathcal{H}^{\|}_\infty$ is given on the dense subspace $\mathcal{H}_\infty \otimes_A \mathcal{H}^{\|}_\infty$by

$$\langle \xi \otimes \eta^{\|}\ | \varsigma\ \otimes\ \rho^{\|} \rangle := \langle \xi_{\mathcal{A}}\, (\eta^{\|} \mid \rho^{\|}) \mid \varsigma \rangle = \langle \xi(\eta|\rho)\mathcal{A} \mid \varsigma \rangle$$

$$= \mathrm{Tr}_\omega((\xi(\eta|\rho)_{\mathcal{A}}|\varsigma)_{\mathcal{A}}\ \langle \mathcal{D} \rangle^{-p}) = \mathrm{Tr}_\omega((\eta|\rho)_{\mathcal{A}}(\xi|\varsigma)_{\mathcal{A}}\ \langle \mathcal{D} \rangle^{-p})$$

for $\xi, \eta, \varsigma, \rho \in \mathcal{H}_\infty$. Hence, the theorem is proved.

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