International Journal of Mathematics Research.
ISSN 0976-5840 Volume 5, Number 1 (2013), pp. 65-76

http://www.irphouse.com

Some Relations on Sixth Order Mock Theta Functions

Roselin Antony

Department of Mathematics, College of Natural and Computational Sciences, P.O. Box 231, Mekelle University, Mekelle, Ethiopia
E-mail: roselinmaths@gmail.com

Abstract

During the last years of his life, Ramanujan defined 17 functions F(q), where $|q| < 1$. Ramanujan named them as mock theta functions, because as q radially approaches any point $e^{2\Pi i\Gamma}$ (r rational), there is a theta function $F_r(q)$ with $F(q) - F_r(q) = 0$ (1). In this paper, we obtain relations connecting mock theta functions, partial mock theta functions of order 6 and infinite products analogous to the identities of Ramanujan.
Keywords: Mock theta functions, partial mock theta functions

1. Preliminaries and Known Results

The first detailed description of mock theta functions was given by Watson in his celebrated Presidential Address delivered at the meeting of the London Mathematical Society in November, 1935.

Ramanujan's general definition of a mock theta function is a function of $f(q)$ defined by a q-series convergent when $|q| < 1$ which satisfies the following two conditions,

(a) For every root ξ of unity, there exist a θ-function θ(q) such that difference between $f(q)$ and θ(q) is bounded as q $\rightarrow$ ξ, radially.

(b) There is no single theta function which works for all ξ, i.e. for every θ-function θ(q) there is some root of unity ξ for which $f(q)$ minus the theta function θ(q) is unbounded as q $\rightarrow$ ξ radially.

Ramanujan gave a list of seventeen mock theta functions and labeled them as third, fifth and seventh orders without giving any reason for his classification. A study of these sums and expansions has been made by Watson (1), Agarwal (2) and Andrews (3). Later on, Andrews and Hickerson (4), Choi (5) and Gordon and Mc Intosh (6) studied certain q-series in the Lost Notebook and named them as sixth, eighth and tenth order mock theta functions. Although Gordon and Mc Intosh (6) have given definitions of order of mock theta functions, and later Bringmann and Ono (7 & 8) have given clarification for the order of the mock theta functions.

Also, relations connecting mock theta functions and partial mock theta functions are given by Srivastava (9) and Denis *et al.* (10). Bhaskar Srivastava (11 & 12) provided relations connecting mock theta functions and partial mock theta functions of order 3, 5, 6 and 10 and relations connecting mock theta functions, partial mock theta functions of order 2, 3 and 6 and Ramanujan's function μ(q) .Recently, Roselin Antony and Atakalti Araya(13) obtained relations connecting mock theta functions of order 2 and infinite products analogous to the identities of Ramanujan.Also,Roselin Antony and Hailemariam Fiseha(14) obtained relations connecting mock theta functions of order 10 and infinite products analogous to the identities of Ramanujan.

If $$M(q)=\sum_{n=0}^{\infty}\Omega_n \tag{1.1}$$

is a mock theta function, then the corresponding partial mock theta function is denoted by the terminating series,

$$M_r(q)=\sum_{n=0}^{\infty}\Omega_n \tag{1.2}$$

Mock theta functions of order 6 ;

In Ramanujan's lost notebook VII, Andrews and Hickerson defined the mock theta functions of order 6 as follows;

$$\phi(q)=\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2}(q;q^2)_n}{(-q;q)_{2n}} \tag{1.3}$$

$$\psi(q)=\sum_{n=0}^{\infty}\frac{(-1)^n q^{(n+1)^2}(q;q^2)_n}{(-q;q)_{2n+1}} \tag{1.4}$$

$$\rho(q)=\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}(-q;q)_n}{(q;q^2)_{n+1}} \tag{1.5}$$

$$\sigma(q)=\sum_{n=0}^{\infty}\frac{q^{n(n+1)(n+2)/2}(-q;q)_n}{(q;q^2)_{n+1}} \tag{1.6}$$

$$\lambda(q)=\sum_{n=0}^{\infty}\frac{(-1)^n q^n(q;q^2)_n}{(-q;q)_n} \tag{1.7}$$

$$2\mu(q)=\sum_{n=0}^{\infty}\frac{(-1)^n q^{n+1}(1+q^n)(q;q^2)_n}{(-q;q)_{n+1}} \tag{1.8}$$

$$\gamma(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}(q;q)_n}{(q^3;q^3)_n} \tag{1.9}$$

Ramanujan, in chapter 16 of his second notebook defined theta functions as follows;

$$A(q)=\sum_{n=0}^{\infty}q^{n(n+1)/2}=\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \tag{1.10}$$

[Ramanujan (15)] and [Berndt (16)]

An identity due to Euler is,

$$\sum_{n=0}^{\infty}\frac{q^{(n+1)^2}}{(q;q^2)_{n+1}}=(-x;q)_\infty \tag{1.11}$$

[Euler (17); chap. 16] and [Andrews (18); Eqn.(2.2.6)]

The special cases of the above identity are ;

$$B(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q^2;q^2)_n}=\frac{(q^2,q^2,q^2;q^4)_\infty}{(q;q)_\infty} \tag{1.12}$$

$$C(q)=\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(q^2;q^2)_n}=\frac{(q,q^3,q^4;q^4)_\infty}{(q;q)_\infty} \tag{1.13}$$

The Famous Roger's –Ramanujan identity is,

$$D(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n}=\frac{1}{(q,q^4;q^5)_\infty} \tag{1.14}$$

$$E(q)=\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(q;q)_n}=\frac{1}{(q^2,q^3;q^5)_\infty} \tag{1.15}$$

[Rogers (19)] and [Ramanujan (20)]

Hahn (21) and Hahn (22) defined the septic analogue of the Rogers- Ramanujan functions as

$$F(q)=\sum_{n=0}^{\infty}\frac{q^{2n^2}}{(q^2;q^2)_n(-q;q)_{2n}}=\frac{(q^3,q^4,q^7;q^7)_\infty}{(q^2;q^2)_\infty} \tag{1.16}$$

$$G(q)=\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{(q^2;q^2)_n(-q;q)_{2n}}=\frac{(q^2,q^5,q^7;q^7)_\infty}{(q^2;q^2)_\infty} \tag{1.17}$$

$$H(q)=\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{(q^2;q^2)_n(-q;q)_{2n+1}}=\frac{(q,q^6,q^7;q^7)_\infty}{(q^2;q^2)_\infty} \tag{1.18}$$

The Jackson – Slater identity;

Jackson (23) discovered the following identity;

$$I(q)=\sum_{n=0}^{\infty}\frac{q^{2n^2}}{(q;q)_{2n}}=\frac{(-q^3,-q^5,q^8;q^8)_\infty}{(q^2;q^2)_\infty} \tag{1.19}$$

This identity was independently rediscovered by Slater (24, Eqn.39) who also discovered its companion identity[Slater (24, Eqn.38)]

$$J(q)=\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{(q;q)_{2n+1}}=\frac{(-q,-q^7,q^8;q^8)_\infty}{(q^2;q^2)_\infty} \tag{1.20}$$

The identity analogous to the Rogers- Ramanujan identity is the so-called Gollnitz – Gordon identity given by,

$$K(q)=\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2}}{(q^2;q^2)_n}=\frac{1}{(q,q^4,q^7;q^8)_\infty} \tag{1.21}$$

$$L(q)=\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2+2n}}{(q^2;q^2)_n}=\frac{1}{(q^3,q^4,q^5;q^8)_\infty} \tag{1.22}$$

[Gordon (25)] and [Gollnitz (26)]

The nonic analogue of Rogers – Ramanujan functions is

$$M(q)=\sum_{n=0}^{\infty}\frac{(q;q)_{3n}.q^{3n^2}}{(q^3;q^3)_n(q^3;q^3)_{2n}}=\frac{(q^4,q^5,q^9;q^9)_\infty}{(q^3;q^3)_\infty} \tag{1.23}$$

$$N(q)=\sum_{n=0}^{\infty}\frac{(q;q)_{3n}(1-q^{3n+2})q^{3n(n+1)}}{(q^3;q^3)_n(q^3;q^3)_{2n+1}}=\frac{(q^2,q^7,q^9;q^9)_\infty}{(q^3;q^3)_\infty} \tag{1.24}$$

$$P(q)=\sum_{n=0}^{\infty}\frac{(q;q)_{3n+1}.q^{3n(n+1)}}{(q^3;q^3)_n(q^3;q^3)_{2n+1}}=\frac{(q,q^8,q^9;q^9)_\infty}{(q^3;q^3)_\infty} \tag{1.25}$$

These equalities are due to Bailey [Bailey (27); Eqn.(1.6),(1.7) and (1.8)].

We shall make use of the following known identity;

$$\sum_{m=0}^{\infty}\delta_m\sum_{r=0}^{m}\alpha_r=\left(\sum_{r=0}^{\infty}\alpha_r\right)\left(\sum_{m=0}^{\infty}\delta_m\right)-\sum_{r=0}^{\infty}\alpha_{r+1}\sum_{m=0}^{r}\delta_m \quad \text{[Srivastava(28) (Eqn. 4.4)]} \tag{1.26}$$

2. Main Results

We shall establish relations connecting mock theta functions, partial mock theta functions of order 6.

A) Taking $\delta_m=q^{m(m+1)/2}$ in (1.26) and by (1.10), we get

$$\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\sum_{r=0}^{\infty}\alpha_r=\sum_{m=0}^{\infty}q^{m(m+1)/2}\sum_{r=0}^{m}\alpha_r+\sum_{r=0}^{\infty}\alpha_{r+1}A_m(q). \tag{2.1}$$

i) Taking $\alpha_r=\dfrac{(-1)^r q^{r^2}(q;q^2)_r}{(-q;q)_{2r}}$ in (2.1) and making use of (1.3), we get

$$\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\phi(q)=\sum_{m=0}^{\infty}q^{m(m+1)/2}\phi_m(q)+\sum_{r=0}^{\infty}\frac{(-1)^{r+1}q^{(r+1)^2}(q;q^2)_{r+1}}{(-q;q)_{2(r+1)}}A_m(q). \tag{2.2}$$

ii) Taking $\alpha_r=\dfrac{(-1)^r q^{(r+1)^2}(q;q^2)_r}{(-q;q)_{2r+1}}$ in (2.1) and making use of (1.4), we get

$$\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\psi(q)=\sum_{m=0}^{\infty}q^{m(m+1)/2}\psi_m(q)+\sum_{r=0}^{\infty}\frac{(-1)^{r+1}q^{(r+2)^2}(q;q^2)_{r+1}}{(-q;q)_{2(r+1)+1}}A_m(q). \tag{2.3}$$

iii) Taking $\alpha_r = \dfrac{q^{r(r+1)/2}(-q;q)_r}{(q;q^2)_{r+1}}$ in (2.1) and making use of (1.5), we get

$$\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\rho(q) = \sum_{m=0}^{\infty} q^{m(m+1)/2}\rho_m(q) + \sum_{r=0}^{\infty}\frac{q^{(r+1)(r+2)/2}(-q;q)_{r+1}}{(q;q^2)_{r+2}}A_m(q). \tag{2.4}$$

iv) Taking $\alpha_r = \dfrac{q^{r(r+1)(r+2)/2}(-q;q)_r}{(q;q^2)_{r+1}}$ in (2.1) and making use of (1.6), we get

$$\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\sigma(q) = \sum_{m=0}^{\infty} q^{m(m+1)/2}\sigma_m(q) + \sum_{r=0}^{\infty}\frac{q^{(r+1)(r+2)(r+3)/2}(-q;q)_{r+1}}{(q;q^2)_{r+2}}A_m(q). \tag{2.5}$$

v) Taking $\alpha_r = \dfrac{(-1)^r q^r (q;q^2)_r}{(-q;q)_r}$ in (2.1) and making use of (1.7), we get

$$\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\lambda(q) = \sum_{m=0}^{\infty} q^{m(m+1)/2}\lambda_m(q) + \sum_{r=0}^{\infty}\frac{(-1)^{r+1}q^{(r+1)}(q;q^2)_{r+1}}{(-q;q)_{r+1}}A_m(q). \tag{2.6}$$

vi) Taking $\alpha_r = \dfrac{(-1)^r q^{r+1}(1+q^r)(q;q^2)_r}{2(-q;q)_{r+1}}$ in (2.1) and making use of (1.8), we get

$$\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\mu(q) = \sum_{m=0}^{\infty} q^{m(m+1)/2}.\mu_m(q) + \sum_{r=0}^{\infty}\frac{(-1)^{r+1}q^{(r+2)}(1+q^r)(q;q^2)_{r+1}}{2(-q;q)_{r+2}}A_m(q). \tag{2.7}$$

vii) Taking $\alpha_r = \dfrac{q^{r^2}(q;q)_r}{(q^3;q^3)_{r+1}}$ in (2.1) and making use of (1.9), we get

$$\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\gamma(q) = \sum_{m=0}^{\infty} q^{m(m+1)/2}.\gamma_m(q) + \sum_{r=0}^{\infty}\frac{(-1)^{r+1}q^{(r+2)}(1+q^r)(q;q^2)_{r+1}}{2(-q;q)_{r+2}}A_m(q). \tag{2.8}$$

B) Taking $\delta_m = \dfrac{q^{m^2}}{(q^2;q^2)_m}$ in (1.26) and by (1.12), we get

$$\frac{(q^2,q^2,q^2;q^4)_\infty}{(q;q)_\infty}\sum_{r=0}^{\infty}\alpha_r = \sum_{m=0}^{\infty}\frac{q^{m^2}}{(q^2;q^2)_m}\sum_{r=0}^{m}\alpha_r + \sum_{r=0}^{\infty}\alpha_{r+1}B_m(q). \tag{2.9}$$

i) Taking $\alpha_r = \dfrac{(-1)^r q^{r^2}(q;q^2)_r}{(-q;q)_{2r}}$ in (2.9) and making use of (1.3), we get

$$\frac{(q^2,q^2,q^2;q^4)_\infty}{(q;q)_\infty}\phi(q) = \sum_{m=0}^{\infty}\frac{q^{m^2}}{(q^2;q^2)_m}\phi_m(q) + \sum_{r=0}^{\infty}\frac{(-1)^{r+1}q^{(r+1)^2}(q;q^2)_{r+1}}{(-q;q)_{2(r+1)}}B_m(q). \quad 2.10)$$

ii) Taking $\alpha_r = \dfrac{(-1)^r q^{(r+1)^2}(q;q^2)_r}{(-q;q)_{2r+1}}$ in (2.9) and making use of (1.4), we get

$$\frac{(q^2,q^2,q^2;q^4)_\infty}{(q;q)_\infty}\psi(q)=\sum_{m=0}^{\infty}\frac{q^{m^2}}{(q^2;q^2)_m}\psi_m(q)+\sum_{r=0}^{\infty}\frac{(-1)^{r+1}q^{(r+2)^2}(q;q^2)_{r+1}}{(-q;q)_{2(r+1)+1}}B_m(q). \quad (2.11)$$

iii) Taking $\alpha_r=\dfrac{q^{r(r+1)/2}(-q;q)_r}{(q;q^2)_{r+1}}$ in (2.9) and making use of (1.5), we get

$$\frac{(q^2,q^2,q^2;q^4)_\infty}{(q;q)_\infty}\rho(q)=\sum_{m=0}^{\infty}\frac{q^{m^2}}{(q^2;q^2)_m}\rho_m(q)+\sum_{r=0}^{\infty}\frac{q^{(r+1)(r+2)/2}(-q;q)_{r+1}}{(q;q^2)_{r+2}}B_m(q). \quad (2.12)$$

iv) Taking $\alpha_r=\dfrac{q^{r(r+1)(r+2)/2}(-q;q)_r}{(q;q^2)_{r+1}}$ in (2.9) and making use of (1.6), we get

$$\frac{(q^2,q^2,q^2;q^4)_\infty}{(q;q)_\infty}\sigma(q)=\sum_{m=0}^{\infty}\frac{q^{m^2}}{(q^2;q^2)_m}\sigma_m(q)+\sum_{r=0}^{\infty}\frac{q^{(r+1)(r+2)(r+3)/2}(-q;q)_{r+1}}{(q;q^2)_{r+2}}B_m(q). \quad (2.13)$$

v) Taking $\alpha_r=\dfrac{(-1)^r q^r(q;q^2)_r}{(-q;q)_r}$ in (2.9) and making use of (1.7), we get

$$\frac{(q^2,q^2,q^2;q^4)_\infty}{(q;q)_\infty}\lambda(q)=\sum_{m=0}^{\infty}\frac{q^{m^2}}{(q^2;q^2)_m}\lambda_m(q)+\sum_{r=0}^{\infty}\frac{(-1)^{r+1}q^{(r+1)}(q;q^2)_{r+1}}{(-q;q)_{r+1}}B_m(q). \quad (2.14)$$

vi) Taking $\alpha_r=\dfrac{(-1)^r q^{r+1}(1+q^r)(q;q^2)_r}{2(-q;q)_{r+1}}$ in (2.9) and making use of (1.8), we get

$$\frac{(q^2,q^2,q^2;q^4)_\infty}{(q;q)_\infty}\mu(q)=\sum_{m=0}^{\infty}\frac{q^{m^2}}{(q^2;q^2)_m}.\mu_m(q)+\sum_{r=0}^{\infty}\frac{(-1)^{r+1}q^{(r+2)}(1+q^r)(q;q^2)_{r+1}}{2(-q;q)_{r+2}}B_m(q). \quad (2.15)$$

vii) Taking $\alpha_r=\dfrac{q^{r^2}(q;q)_r}{(q^3;q^3)_{r+1}}$ in (2.9) and making use of (1.9), we get

$$\frac{(q^2,q^2,q^2;q^4)_\infty}{(q;q)_\infty}\gamma(q)=\sum_{m=0}^{\infty}\frac{q^{m^2}}{(q^2;q^2)_m}.\gamma_m(q)+\sum_{r=0}^{\infty}\frac{(-1)^{r+1}q^{(r+2)}(1+q^r)(q;q^2)_{r+1}}{2(-q;q)_{r+2}}B_m(q). \quad (2.16)$$

Similarly, by taking $\delta_m=\dfrac{q^{m(m+1)}}{(q^2;q^2)_m}$, we can establish relations connecting mock theta functions of order 6 and the infinite product $C(q)$.

C) Taking $\delta_m=\dfrac{q^{m^2}}{(q;q)_m}$ in (1.26) and by (1.14), we get

$$\frac{1}{(q,q^4;q^5)_\infty}\sum_{r=0}^{\infty}\alpha_r=\sum_{m=0}^{\infty}\frac{q^{m^2}}{(q;q)_m}\sum_{r=0}^{m}\alpha_r+\sum_{r=0}^{\infty}\alpha_{r+1}D_m(q). \quad (2.17)$$

i) Taking $\alpha_r=\dfrac{(-1)^r q^{r^2}(q;q^2)_r}{(-q;q)_{2r}}$ in (2.17) and making use of (1.3), we get

$$\frac{1}{(q,q^4;q^5)_\infty}\phi(q)=\sum_{m=0}^{\infty}\frac{q^{m^2}}{(q;q)_m}\phi_m(q)+\sum_{r=0}^{\infty}\frac{(-1)^{r+1}q^{(r+1)^2}(q;q^2)_{r+1}}{(-q;q)_{2(r+1)}}D_m(q). \quad (2.18)$$

ii) Taking $\alpha_r=\dfrac{(-1)^r q^{(r+1)^2}(q;q^2)_r}{(-q;q)_{2r+1}}$ in (2.17) and making use of (1.4), we get

$$\frac{1}{(q,q^4;q^5)_\infty}\psi(q)=\sum_{m=0}^{\infty}\frac{q^{m^2}}{(q;q)_m}\psi_m(q)+\sum_{r=0}^{\infty}\frac{(-1)^{r+1}q^{(r+2)^2}(q;q^2)_{r+1}}{(-q;q)_{2(r+1)+1}}D_m(q). \quad (2.19)$$

iii) Taking $\alpha_r=\dfrac{q^{r(r+1)/2}(-q;q)_r}{(q;q^2)_{r+1}}$ in (2.17) and making use of (1.5), we get

$$\frac{1}{(q,q^4;q^5)_\infty}\rho(q)=\sum_{m=0}^{\infty}\frac{q^{m^2}}{(q;q)_m}\rho_m(q)+\sum_{r=0}^{\infty}\frac{q^{(r+1)(r+2)/2}(-q;q)_{r+1}}{(q;q^2)_{r+2}}D_m(q). \quad (2.20)$$

iv) Taking $\alpha_r=\dfrac{q^{r(r+1)(r+2)/2}(-q;q)_r}{(q;q^2)_{r+1}}$ in (2.17) and making use of (1.6), we get

$$\frac{1}{(q,q^4;q^5)_\infty}\sigma(q)=\sum_{m=0}^{\infty}\frac{q^{m^2}}{(q;q)_m}\sigma_m(q)+\sum_{r=0}^{\infty}\frac{q^{(r+1)(r+2)(r+3)/2}(-q;q)_{r+1}}{(q;q^2)_{r+2}}D_m(q). \quad (2.21)$$

v) Taking $\alpha_r=\dfrac{(-1)^r q^r(q;q^2)_r}{(-q;q)_r}$ in (2.17) and making use of (1.7), we get

$$\frac{1}{(q,q^4;q^5)_\infty}\lambda(q)=\sum_{m=0}^{\infty}\frac{q^{m^2}}{(q;q)_m}\lambda_m(q)+\sum_{r=0}^{\infty}\frac{(-1)^{r+1}q^{(r+1)}(q;q^2)_{r+1}}{(-q;q)_{r+1}}D_m(q). \quad (2.22)$$

vi) Taking $\alpha_r=\dfrac{(-1)^r q^{r+1}(1+q^r)(q;q^2)_r}{2(-q;q)_{r+1}}$ in (2.17) and making use of (1.8), we get

$$\frac{1}{(q,q^4;q^5)_\infty}\mu(q)=\sum_{m=0}^{\infty}\frac{q^{m^2}}{(q;q)_m}.\mu_m(q)+\sum_{r=0}^{\infty}\frac{(-1)^{r+1}q^{(r+2)}(1+q^r)(q;q^2)_{r+1}}{2(-q;q)_{r+2}}D_m(q).$$

(2.23)

vii) Taking $\alpha_r=\dfrac{q^{r^2}(q;q)_r}{(q^3;q^3)_{r+1}}$ in (2.17) and making use of (1.9), we get

$$\frac{1}{(q,q^4;q^5)_\infty}\gamma(q)=\sum_{m=0}^{\infty}\frac{q^{m^2}}{(q;q)_m}\gamma_m(q)+\sum_{r=0}^{\infty}\frac{(-1)^{r+1}q^{(r+2)}(1+q^r)(q;q^2)_{r+1}}{2(-q;q)_{r+2}}D_m(q). \quad (2.24)$$

Again, taking $\delta_m=\dfrac{q^{m(m+1)}}{(q;q)_m}$, relations can be developed connecting mock theta functions and the infinite product $E(q)$.

D) Taking $\delta_m = \dfrac{q^{2m^2}}{(q^2;q^2)_m(-q;q)_{2m}}$ in (1.26) and by (1.16), we get

$$\frac{(q^3,q^4,q^7;q^7)_\infty}{(q^2;q^2)_\infty}\sum_{r=0}^{\infty}\alpha_r = \sum_{m=0}^{\infty}\frac{q^{2m^2}}{(q^2;q^2)_m(-q;q)_{2m}}\sum_{r=0}^{m}\alpha_r + \sum_{r=0}^{\infty}\alpha_{r+1}F_m(q). \tag{2.25}$$

i) Taking $\alpha_r = \dfrac{(-1)^r q^{r^2}(q;q^2)_r}{(-q;q)_{2r}}$ in (2.25) and making use of (1.3), we get

$$\frac{(q^3,q^4,q^7;q^7)_\infty}{(q^2;q^2)_\infty}\phi(q) = \sum_{m=0}^{\infty}\frac{q^{2m^2}}{(q^2;q^2)_m(-q;q)_{2m}}\phi_m(q) + \sum_{r=0}^{\infty}\frac{(-1)^{r+1}q^{(r+1)^2}(q;q^2)_{r+1}}{(-q;q)_{2(r+1)}}F_m(q). \tag{2.26}$$

ii) Taking $\alpha_r = \dfrac{(-1)^r q^{(r+1)^2}(q;q^2)_r}{(-q;q)_{2r+1}}$ in (2.25) and making use of (1.4), we get

$$\frac{(q^3,q^4,q^7;q^7)_\infty}{(q^2;q^2)_\infty}\psi(q) = \sum_{m=0}^{\infty}\frac{q^{2m^2}}{(q^2;q^2)_m(-q;q)_{2m}}\psi_m(q) + \sum_{r=0}^{\infty}\frac{(-1)^{r+1}q^{(r+2)^2}(q;q^2)_{r+1}}{(-q;q)_{2(r+1)+1}}F_m(q). \tag{2.27}$$

iii) Taking $\alpha_r = \dfrac{q^{r(r+1)/2}(-q;q)_r}{(q;q^2)_{r+1}}$ in (2.25) and making use of (1.5), we get

$$\frac{(q^3,q^4,q^7;q^7)_\infty}{(q^2;q^2)_\infty}\rho(q) = \sum_{m=0}^{\infty}\frac{q^{2m^2}}{(q^2;q^2)_m(-q;q)_{2m}}\rho_m(q) + \sum_{r=0}^{\infty}\frac{q^{(r+1)(r+2)/2}(-q;q)_{r+1}}{(q;q^2)_{r+2}}F_m(q). \tag{2.28}$$

iv) Taking $\alpha_r = \dfrac{q^{r(r+1)(r+2)/2}(-q;q)_r}{(q;q^2)_{r+1}}$ in (2.25) and making use of (1.6), we get

$$\frac{(q^3,q^4,q^7;q^7)_\infty}{(q^2;q^2)_\infty}\sigma(q) = \sum_{m=0}^{\infty}\frac{q^{2m^2}}{(q^2;q^2)_m(-q;q)_{2m}}\sigma_m(q) + \sum_{r=0}^{\infty}\frac{q^{(r+1)(r+2)(r+3)/2}(-q;q)_{r+1}}{(q;q^2)_{r+2}}F_m(q). \tag{2.29}$$

v) Taking $\alpha_r = \dfrac{(-1)^r q^r(q;q^2)_r}{(-q;q)_r}$ in (2.25) and making use of (1.7), we get

$$\frac{(q^3,q^4q^7;q^7)_\infty}{(q^2;q^2)_\infty}\lambda(q) = \sum_{m=0}^{\infty}\frac{q^{2m^2}}{(q^2;q^2)_m(-q;q)_{2m}}\lambda_m(q) + \sum_{r=0}^{\infty}\frac{(-1)^{r+1}q^{(r+1)}(q;q^2)_{r+1}}{(-q;q)_{r+1}}F_m(q). \tag{2.30}$$

vi) Taking $\alpha_r = \dfrac{(-1)^r q^{r+1}(1+q^r)(q;q^2)_r}{2(-q;q)_{r+1}}$ in (2.25) and making use of (1.8), we get

$$\frac{(q^3,q^4q^7;q^7)_\infty}{(q^2;q^2)_\infty}\mu(q) = \sum_{m=0}^{\infty}\frac{q^{2m^2}}{(q^2;q^2)_m(-q;q)_{2m}}\mu_m(q) + \sum_{r=0}^{\infty}\frac{(-1)^{r+1}q^{(r+2)}(1+q^r)(q;q^2)_{r+1}}{2(-q;q)_{r+2}}F_m(q). \tag{2.31}$$

vii) Taking $\alpha_r = \dfrac{q^{r^2}(q;q)_r}{(q^3;q^3)_{r+1}}$ in (2.25) and making use of (1.9), we get

$$\frac{(q^3,q^4,q^7;q^7)_\infty}{(q,q^4;q^5)_\infty}\gamma(q)=\sum_{m=0}^{\infty}\frac{q^{2m^2}}{(q^2;q^2)_m(-q;q)_{2m}}\gamma_m(q)+\sum_{r=0}^{\infty}\frac{(-1)^{r+1}q^{(r+2)}(1+q^r)(q;q^2)_{r+1}}{2(-q;q)_{r+2}}F_m(q). \tag{2.32}$$

In the similar way, by assuming $\delta_m = \dfrac{q^{2m(m+1)}}{(q^2;q^2)_m(-q;q)_{2m}}$ and

$\delta_m = \dfrac{q^{2m(m+1)}}{(q^2;q^2)_m(-q;q)_{2m+1}}$, relations connecting mock theta functions of order six and the infinite products $G(q)$ and $H(q)$ can be obtained.

E) Taking $\delta_m = \dfrac{q^{2m^2}}{(q;q)_{2m}}$ in (1.26) and by (1.19), we get

$$\frac{(-q^3,-q^5,q^8;q^8)_\infty}{(q^2;q^2)_\infty}\sum_{r=0}^{\infty}\alpha_r=\sum_{m=0}^{\infty}\frac{q^{2m^2}}{(q;q)_{2m}}\sum_{r=0}^{m}\alpha_r+\sum_{r=0}^{\infty}\alpha_{r+1}I_m(q). \tag{2.33}$$

i) Taking $\alpha_r = \dfrac{(-1)^r q^{r^2}(q;q^2)_r}{(-q;q)_{2r}}$ in (2.33) and making use of (1.3), we get

$$\frac{(-q^3,-q^5,q^8;q^8)_\infty}{(q^2;q^2)_\infty}\phi(q)=\sum_{m=0}^{\infty}\frac{q^{2m^2}}{(q;q)_{2m}}\phi_m(q)+\sum_{r=0}^{\infty}\frac{(-1)^{r+1}q^{(r+1)^2}(q;q^2)_{r+1}}{(-q;q)_{2(r+1)}}I_m(q). \tag{2.34}$$

ii) Taking $\alpha_r = \dfrac{(-1)^r q^{(r+1)^2}(q;q^2)_r}{(-q;q)_{2r+1}}$ in (2.33) and making use of (1.4), we get

$$\frac{(-q^3,-q^5,q^8;q^8)_\infty}{(q^2;q^2)_\infty}\psi(q)=\sum_{m=0}^{\infty}\frac{q^{2m^2}}{(q;q)_{2m}}\psi_m(q)+\sum_{r=0}^{\infty}\frac{(-1)^{r+1}q^{(r+2)^2}(q;q^2)_{r+1}}{(-q;q)_{2(r+1)+1}}I_m(q). \tag{2.35}$$

iii) Taking $\alpha_r = \dfrac{q^{r(r+1)/2}(-q;q)_r}{(q;q^2)_{r+1}}$ in (2.33) and making use of (1.5), we get

$$\frac{(-q^3,-q^5,q^8;q^8)_\infty}{(q^2;q^2)_\infty}\rho(q)=\sum_{m=0}^{\infty}\frac{q^{2m^2}}{(q;q)_{2m}}\rho_m(q)+\sum_{r=0}^{\infty}\frac{q^{(r+1)(r+2)/2}(-q;q)_{r+1}}{(q;q^2)_{r+2}}I_m(q). \tag{2.36}$$

iv) Taking $\alpha_r = \dfrac{q^{r(r+1)(r+2)/2}(-q;q)_r}{(q;q^2)_{r+1}}$ in (2.33) and making use of (1.6), we get

$$\frac{(-q^3,-q^5,q^8;q^8)_\infty}{(q^2;q^2)_\infty}\sigma(q)=\sum_{m=0}^{\infty}\frac{q^{2m^2}}{(q;q)_{2m}}\sigma_m(q)+\sum_{r=0}^{\infty}\frac{q^{(r+1)(r+2)(r+3)/2}(-q;q)_{r+1}}{(q;q^2)_{r+2}}I_m(q). \tag{2.37}$$

v) Taking $\alpha_r = \dfrac{(-1)^r q^r (q;q^2)_r}{(-q;q)_r}$ in (2.33) and making use of (1.7), we get

$$\frac{(-q^3,-q^5,q^8;q^8)_\infty}{(q^2;q^2)_\infty}\lambda(q) = \sum_{m=0}^{\infty}\frac{q^{2m^2}}{(q;q)_{2m}}\lambda_m(q) + \sum_{r=0}^{\infty}\frac{(-1)^{r+1}q^{(r+1)}(q;q^2)_{r+1}}{(-q;q)_{r+1}}I_m(q). \quad (2.38)$$

vi) Taking $\alpha_r = \dfrac{(-1)^r q^{r+1}(1+q^r)(q;q^2)_r}{2(-q;q)_{r+1}}$ in (2.33) and making use of (1.8), we get

$$\frac{(-q^3,-q^5,q^8;q^8)_\infty}{(q^2;q^2)_\infty}\mu(q) = \sum_{m=0}^{\infty}\frac{q^{2m^2}}{(q;q)_{2m}}.\mu_m(q) + \sum_{r=0}^{\infty}\frac{(-1)^{r+1}q^{(r+2)}(1+q^r)(q;q^2)_{r+1}}{2(-q;q)_{r+2}}I_m(q). \quad (2.39)$$

vii) Taking $\alpha_r = \dfrac{q^{r^2}(q;q)_r}{(q^3;q^3)_{r+1}}$ in (2.33) and making use of (1.9), we get

$$\frac{(-q^3,-q^5,q^8;q^8)_\infty}{(q^2;q^2)_\infty}\gamma(q) = \sum_{m=0}^{\infty}\frac{q^{2m^2}}{(q;q)_{2m}}.\gamma_m(q) + \sum_{r=0}^{\infty}\frac{(-1)^{r+1}q^{(r+2)}(1+q^r)(q;q^2)_{r+1}}{2(-q;q)_{r+2}}I_m(q). \quad (2.40)$$

In the similar way, by assuming $\delta_m = \dfrac{q^{2m(m+1)}}{(q;q)_{2m+1}}, \delta_m = \dfrac{(-q;q^2)_m q^{m^2}}{(q^2;q^2)_m}$

and $\delta_m = \dfrac{(-q;q^2)_m q^{m^2+2m}}{(q^2;q^2)_m}$, relations connecting mock theta functions of order six and the infinite products $J(q)$, $K(q)$ and $L(q)$ can be obtained.

F) Taking $\delta_m = \dfrac{(q;q)_{3n} q^{3m^2}}{(q^3:q^3)_m (q^3;q^3)_{2m}}$ in (1.26) and by (1.23), we get

$$\frac{(q^4,q^5,q^9;q^9)_\infty}{(q^3;q^3)_\infty}\sum_{r=0}^{\infty}\alpha_r = \sum_{m=0}^{\infty}\frac{(q;q)_{3m}q^{3m^2}}{(q^3;q^3)_m(q^3;q^3)_{2m}}\sum_{r=0}^{m}\alpha_r + \sum_{r=0}^{\infty}\alpha_{r+1}M_m(q). \quad (2.41)$$

i) Taking $\alpha_r = \dfrac{(-1)^r q^{r^2}(q;q^2)_r}{(-q;q)_{2r}}$ in (2.41) and making use of (1.3), we get

$$\frac{(q^4,q^5,q^9;q^9)_\infty}{(q^3;q^3)_\infty}\phi(q) = \sum_{m=0}^{\infty}\frac{(q;q)_{3m}q^{3m^2}}{(q^3;q^3)_m(q^3;q^3)_{2m}}\phi_m(q) + \sum_{r=0}^{\infty}\frac{(-1)^{r+1}q^{(r+1)^2}(q;q^2)_{r+1}}{(-q;q)_{2(r+1)}}M_m(q). \quad (2.42)$$

ii) Taking $\alpha_r = \dfrac{(-1)^r q^{(r+1)^2}(q;q^2)_r}{(-q;q)_{2r+1}}$ in (2.41) and making use of (1.4), we get

$$\frac{(q^4,q^5,q^9;q^9)_\infty}{(q^3;q^3)_\infty}\psi(q) = \sum_{m=0}^{\infty}\frac{(q;q)_{3m}q^{3m^2}}{(q^3;q^3)_m(q^3;q^3)_{2m}}\psi_m(q) + \sum_{r=0}^{\infty}\frac{(-1)^{r+1}q^{(r+2)^2}(q;q^2)_{r+1}}{(-q;q)_{2(r+1)+1}}M_m(q). \quad (2.43)$$

iii) Taking $\alpha_r = \dfrac{q^{r(r+1)/2}(-q;q)_r}{(q;q^2)_{r+1}}$ in (2.41) and making use of (1.5), we get

$$\frac{(q^4,q^5,q^9;q^9)_\infty}{(q^3;q^3)_\infty}\rho(q) = \sum_{m=0}^{\infty}\frac{(q;q)_{3m}q^{3m^2}}{(q^3;q^3)_m(q^3;q^3)_{2m}}\rho_m(q) + \sum_{r=0}^{\infty}\frac{q^{(r+1)(r+2)/2}(-q;q)_{r+1}}{(q;q^2)_{r+2}}M_m(q). \qquad (2.44)$$

iv) Taking $\alpha_r = \dfrac{q^{r(r+1)(r+2)/2}(-q;q)_r}{(q;q^2)_{r+1}}$ in (2.41) and making use of (1.6), we get

$$\frac{(q^4,q^5,q^9;q^9)_\infty}{(q^3;q^3)_\infty}\sigma(q) = \sum_{m=0}^{\infty}\frac{(q;q)_{3m}q^{3m^2}}{(q^3;q^3)_m(q^3;q^3)_{2m}}\sigma_m(q) + \sum_{r=0}^{\infty}\frac{q^{(r+1)(r+2)(r+3)/2}(-q;q)_{r+1}}{(q;q^2)_{r+2}}M_m(q). \qquad (2.45)$$

v) Taking $\alpha_r = \dfrac{(-1)^r q^r (q;q^2)_r}{(-q;q)_r}$ in (2.41) and making use of (1.7), we get

$$\frac{(q^4,q^5,q^9;q^9)_\infty}{(q^3;q^3)_\infty}\lambda(q) = \sum_{m=0}^{\infty}\frac{(q;q)_{3m}q^{3m^2}}{(q^3;q^3)_m(q^3;q^3)_{2m}}\lambda_m(q) + \sum_{r=0}^{\infty}\frac{(-1)^{r+1}q^{(r+1)}(q;q^2)_{r+1}}{(-q;q)_{r+1}}M_m(q). \qquad (2.46)$$

vi) Taking $\alpha_r = \dfrac{(-1)^r q^{r+1}(1+q^r)(q;q^2)_r}{2(-q;q)_{r+1}}$ in (2.41) and making use of (1.8), we get

$$\frac{(q^4,q^5,q^9;q^9)_\infty}{(q^3;q^3)_\infty}\mu(q) = \sum_{m=0}^{\infty}\frac{(q;q)_{3m}q^{3m^2}}{(q^3;q^3)_m(q^3;q^3)_{2m}}\mu_m(q) + \sum_{r=0}^{\infty}\frac{(-1)^{r+1}q^{(r+2)}(1+q^r)(q;q^2)_{r+1}}{2(-q;q)_{r+2}}M_m(q). \qquad (2.47)$$

vii) Taking $\alpha_r = \dfrac{q^{r^2}(q;q)_r}{(q^3;q^3)_{r+1}}$ in (2.41) and making use of (1.9), we get

$$\frac{(q^4,q^5,q^9;q^9)_\infty}{(q^3;q^3)_\infty}\gamma(q) = \sum_{m=0}^{\infty}\frac{(q;q)_{3m}q^{3m^2}}{(q^3;q^3)_m(q^3;q^3)_{2m}}\gamma_m(q) + \sum_{r=0}^{\infty}\frac{(-1)^{r+1}q^{(r+2)}(1+q^r)(q;q^2)_{r+1}}{2(-q;q)_{r+2}}M_m(q). \qquad (2.48)$$

In the same way, by assuming

$\delta_m = \dfrac{(q;q)_{3m}(1-q^{3m+2})q^{3m(m+1)}}{(q^3;q^3)_m(q^3;q^3)_{2m+1}}$ and $\delta_m = \dfrac{(q;q)_{3m+1}q^{3m+1}}{(q^3;q^3)_m(q^3;q^3)_{2m+1}}$, relations connecting mock theta functions of order six and the infinite products $N(q)$ and $P(q)$ can be obtained.

References

[1] Watson, G.N., 1936, "The final problem: an account of the mock theta functions," Jour. London Math. Soc., 11, pp. 55-80.

[2] Agarwal, R.P., 1969, "Certain basic hypergeometric identities associated with mock theta functions," Quart. Jour. Math., 20, pp. 121-128.

[3] Andrews, G.E., 1981, "Ramanujan's lost notebook – I : Partial (-) functions," Adv. Math., 41, pp. 137-170.

[4] Andrews, G.E. and Hickerson, D., 1991, "Ramanujan's lost notebook - III : The sixth order mock theta functions," Adv. Math., 89, pp. 60-105.

[5] Choi, Y.S., 1999, "Tenth order mock theta functions in Ramanujan's lost notebook," Invent. Math., 136, pp. 497-596.
[6] Gordon, B. and Mc Intosh, R.J., 2000, "Some eighth order mock theta functions," Jour. Lodon Math. Soc., 62, pp. 321-335.
[7] Bringmann, K., Ono, K., "The f(q) mock theta functions conjecture and partition ranks," Inven. Math., 165, pp. 243.
[8] Bringmann, K., Ono, K., 2007, "Lifting elliptic cups forms to maass form with an application to partitions," Proc. Math. Acad. Sci. (USA), 104, pp. 3725.
[9] Srivastava, A.K., 1997, "On Partial sums of mock theta functions of order three," Proc. Indian Aca. Sci. (Math. Sci.), 107 (1), pp. 1-12.
[10] Denis, R.Y., Singh, S.N. and Singh, S.P., 2006, "On certain relation connecting mock theta functions," Italian Jour. Pure & Appl. Math., 19, pp. 55-60.
[11] Srivastava, B., 2005, "Ramanujan's mock theta functions," Math. J. Okayama Univ. 47(1), pp. 163 – 174.
[12] ________, 2007, "Partial second order mock theta functions their expansions and pade approximations," J. Korean Math. Soc., 44(4), pp. 767 – 777.
[13] Antony, R. and Araya, A., 2013, "On partial sums of Mock theta functions of order 2," International journal of Advanced computer and Mathematical sciences, 4(1), pp. 586 – 603.
[14] Roselin Antony, R. and Fiseha, H., 2012, "On certain relations connecting mock theta functions of order 10," International Journal of Engineering and Mathematical Sciences, 2(1), pp. 83-101.
[15] Ramanujan, S., 1957, "Ramanujan's Notebooks (Vols. I and II)," Tata Institute of Fundamental Research, Bombay.
[16] Berndt, B.C., 1991, "Ramanujan's Notebooks - Part III," Springer, New York.
[17] Euler, L., 1748, "Introductio in Analysin Infinitorum," Marcum – Michaelem Bousuet, Lausannae.
[18] Andrews, G.E., 1988, "The theory of partitions; Encyclopedia of Mathematics and its Applications," No. 2., (Addison – Wesley, 1976; reissued by Camb. Univ. Press, 1998).
[19] Rogers, L.J., 1894, "Second memoir on the expansion of certain infinite products," Proc. London Math. Soc., 25, pp. 318 - 343.
[20] Ramanujan, S., 1919, "Proof of certain identities in combinatory analysis," Proc. Camb. Philos. Soc., 19, pp. 214 - 216.
[21] Hahn, H., 2003, "Septic Analogues of the Rogers-Ramanujan functions," Acta Arith., 110, pp. 381-399.
[22] Hahn, H., 2004, "Einstein series, analogues of the Rogers-Ramanujan functions and partitions," Ph. D. thesis, University of Illinois at Urbana-champaign.
[23] Jackson, F.H., 1928, "Examples of a generalization of Euler's transformation for power series," Messenger of Math., 57, pp. 169 – 187.
[24] Slater, L.J., 1952, "Further identities of the Rogers – Ramanujan type," Proc. London Math. Soc., 54(2), pp. 147 – 167.
[25] Gordon, B., 1965, "Some continued fractions of Rogers-Ramanujan type," Duke Math. J., 32, pp. 741 - 448.
[26] Gollnitz, H., 1967,"Partitionen mit Differenzenbedingunger," J. Reine Angew Math., 225, pp. 154 - 190.
[27] Bailey, W.N., 1947, "Some identities in combinatory analysis," Proc. London Math. Soc., 49, pp. 421 - 435.
[28] Srivastava, A.K., 1997, "On Partial sums of mock theta functions of order three," Proc. Indian Aca. Sci. (Math. Sci.), 107 (1), pp. 1 - 12.