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Cone metric space and Fixed point theorems for pair of generalized contractive mappings

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Abstract

In his paper, we extend and generalize some common fixed point theorems for contractive mappings in cone metric spaces. Our results generalize several well known of recent results.

Keyword: Fixed point, common fixed point, generalized contractive mapping, complete cone metric space, ordered banach space.
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Introduction

Very recently, Huang and Zhang [1], introduced the concept of metric space by replacing the set of real numbers by on ordered Banach space. They prove some fixed point theorems for contractive mappings by using normality of cone. The results in [1] were generalized by Sh. Rezapour and R. Hamlbarani [2] omitting the assumption of normality on the cone. Subsequently many others have generalized the results of Huang and Zhang have studies fixed point theorems for normal and non-normal cone. Rhoades [4] made a comparison of various different type of definition of contractive mappings.
In the present paper, we study the existence of a common fixed point for pair of contractive mappings in the setting of complete cone spaces.

Preliminary notes

First, we recall some standard notations and definitions in cone metric spaces with some of their properties [1].

Definition 2.1 Let E be a real Banach space and P a subset of E. P is called a cone if and only if:

(i) P is closed , non- empty , and P = {0},

(ii) a , b ∈ R, b ≥ 0, x , y ∈ p ⇒ax+by ∈ P ,

(iii) x ∈ P and - x ∈ P ⇒ x = 0.

Given a cone P⊂E, we define a partial ordering ≤ on E with respect to P by $x \le y$ if and only if y- x ∈ P. We shall write $x \ll y$ if y- x ∈ intP , int P denotes the interior of P . The cone P is called the normal if there is a number K > 0 such that for all x , y ∈ E ,

$$0 \le x \le y \text{ implies } \| x \| \le K \| y \|$$

The least positive number K satisfying the above is called the normal constant of P.

Definition 2.2 Let X be a non-empty set and d:X $\boldsymbol{x}$ X ⟶ E a mapping such that

(d_1) 0≤ d(x, y) for all x ,y ∈ X and d (x, y) = 0 if and only if x = y ,

(d_2) d (x, y) = d (x, y) for all x , y ∈ X ,

(d_3) d (x, y) ≤ d (x,z) + d (z ,y) for all x, y, z ∈ X.

Then d is called cone metric on X , and (X ,d) is called a cone metric space[1].

Example 2.3 Let E = R^2 , P = {(x, y) ∈ E : x, y ≥ 0}, x =R and d: X$\boldsymbol{x}$X ⟶ E defined by $d(x, y) = (| x - y| , \alpha | x - y|)$, where $\alpha \ge 0$ is a constant .

Then (X, d) is a cone metric space [1].

Definition 2.4 [1] Let (X, d) be a cone metric space, x∈ X and $\{x_n\}_{n\ge1}$ a sequence in X . Then,

(i) $\{x_n\}_{n\ge1}$ converges to x whenever for every c∈ E with 0≪ c there is a natural number N such that d(x_n ,x) ≪ c for all n≥ N . We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \longrightarrow x(n\longrightarrow\infty)$.

(ii) $\{x_n\}_{n\ge1}$ is said to Cauchy sequence if for every c∈ E with 0≪ c there is a natural number N such that d(x_n,x_m)≪ c for all n , m≥ N.

(iii) (X, d) is called a complete cone metric space if every Cauchy sequence in X is convergent.

Lemma2.5 [1] Let (X, d) be a cone metric space, P⊂E a normal cone with normal constant K. Let $\{x_n\}$, $\{y_n\}$ be sequence in X and x, y∈ X.

(i) $\{x_n\}$ converges to x if and only if limn⟶∞ d(x_n , x) = 0;

(ii) If $\{x_n\}$ converges to x and $\{x_n\}$ converges to y then $x = y$ That is the limit of $\{x_n\}$ is unique;

(iii) If $\{x_n\}$ converges to x , then $\{x_n\}$ is Cauchy sequence ;

(iv) If$\{x_n\}$ is a Cauchy sequence if and only if $\lim_{n, m\to\infty} d(x_n, x_m) = 0$;

(v) If $x_n \longrightarrow$ x and $y_n \longrightarrow$ y then d(x_n, y_n) then $d(x_n,y_n) \longrightarrow d(x,y)$

Definition 2.6 [1] Let (X, d) be a cone metric space, P a normal cone with normal constant K and T: X ⟶X. The

(i) T is said to be continuous if $\lim_{n\to\infty} x_n = x$, implies that $\lim_{n\to\infty} T(x_n) = T(x)$, for all $\{x_n\}$ in X ;

(ii) T is said to be subsequentialy convergent , if we have , for every sequence $\{y_n)$ that T$\{y_n\}$ has a convergent subsequence ;

(iii) T is said to be sequentially convergent, if we have, for every sequence $\{y_n\}$, if T(y_n) is convergent, then $\{y_n\}$ is also convergent .

3. *Main Results*

In this section, we give some results which generalize theorems 2.1, 2.2, 2.3, 2.4 of (6) and so theorems 1, 2, 3, and 4 of (1).

Theorem 3.1 Let (X, d) be a complete cone metric space, P be a normal cone with normal constant. Assume that, Let T: X →X be a continuous and one to one mapping and suppose

R, S: X →X is a pair satisfying the contractive condition

$$d(\mathrm{T\,R}x, \mathrm{TS}y) \leq \mathrm{K\,d}\,(\mathrm{T}x, \mathrm{Ty}) \qquad (3.1.1)$$

Where K ∊ [0, 1) is a constant. Then the pairs T, R and S have a unique common fixed point in X. And for any x ∊ X, iterative sequences $\{\mathrm{TR}^{2n+1} x\}$ and $\{\mathrm{TS}^{2n+2} x\}$ converges to the common fixed point.

Proof: Choose x_0 ∊ X, set $x_1 = \mathrm{T\,R}x_0, x_3 = \mathrm{T\,R}x_2 = \mathrm{T\,R}^3 x_0$---------- $x_{2n+1} = \mathrm{TR}x_{2n} = \mathrm{TR}^{2n+1}x_0$---------

Similarly, we can have $x_2 = \mathrm{TS}x_1 = \mathrm{TS}^2 x_0 x_4 = \mathrm{TS}x_3 = \mathrm{TS}^4 x_0$------------ $x_{2n+1} = \mathrm{TS}x_{2n} =$ -----$=\mathrm{TS}^{2n+2}x_0$----------

We have

$$d(\mathrm{T}x_{2n+1}, \mathrm{T}x_{2n}) = d(\mathrm{TR}x_{2n}, \mathrm{TS}x_{2n-1}) \leq \mathrm{K}d(\mathrm{T}x_{2n}, \mathrm{T}x_{2n-1}) \leq \mathrm{K}^2 d(\mathrm{T}x_{2n-1}, \mathrm{T}x_{2n-2})$$
$$\leq \text{-----------} \leq \mathrm{K}^{2n} d(\mathrm{T}x_1, \mathrm{T}x_0)$$

So, for n>m

$$d(\mathrm{T}x_{2n}, \mathrm{T}x_{2m}) \leq d(\mathrm{T}x_{2n}, \mathrm{T}x_{2n-1}) + d(\mathrm{T}x_{2n-1}, \mathrm{T}x_{2n-2}) + \text{-------------} + d(\mathrm{T}x_{2m+1}, \mathrm{T}x_{2m})$$
$$\leq (\mathrm{K}^{2n-1} + \mathrm{K}^{2n-2} + \text{-------} + \mathrm{K}^{2m})\, d(\mathrm{T}x_1, \mathrm{T}x_0)$$
$$\leq \frac{\mathrm{K}^{2m}}{1-\mathrm{K}}\, d(\mathrm{T}x_1, \mathrm{T}x_0) \qquad (3.1.2)$$

Since P is a normal cone with normal constant K; so, we get

$$\| d(\mathrm{T}x_{2n}, \mathrm{T}x_{2m}) \| \leq \frac{\mathrm{K}^{2m}}{1-\mathrm{K}}\, \mathrm{K}\, \| d(\mathrm{T}x_1, \mathrm{T}x_0) \| \qquad (3.1.3)$$

This implies $d(\mathrm{T}x_{2n}, \mathrm{T}x_{2m}) \to 0$ $(n, m\to\infty)$

Hence $\{\mathrm{T}x_{2n}\}$ is a Cauchy Sequence.
Since (X, d) is a complex cone metric space, there exist $v \in$ X such that $\mathrm{T}x_{2n}$ $\to v$ $(n\to\infty)$ $i.e$ $\lim_{n\to\infty}\mathrm{T}x_{2n} = v$ --------------------- (3.1.4)

If T subsequently convergent, $\{x_{2n}\}$ has a convergent sub sequence. So there exist $u \in$ X and $\{x_{2n(i)}\}$ such that
$\lim_{n\to\infty} x_{2n(i)} = u$.------------ ------------------- (3.1.5)

Since T is continuous and by (3.1.5) we obtain

$\lim_{n\to\infty} \mathrm{T}x_{2n(i)} = \mathrm{T}u$---------------------------- (3.1.6)

By (3.1.4) and (3.1.6) we conclude that

$\mathrm{T}\,u = v$------------------------------ (3.1.7)

On the other hand

$$d(\mathrm{TR}u, \mathrm{T}u) \le d(\mathrm{TR}x_{2n(i)}, \mathrm{TR}\,u) + d(\mathrm{TR}x_{2n(i)}, \mathrm{T}u)$$

$$\le \mathrm{K}(\mathrm{T}x_{2n(i)}, \mathrm{T}u) + d(\mathrm{T}x_{2n+1(i+1)}, \mathrm{T}u)$$

$$\| d(\mathrm{TR}u, \mathrm{T}u) \| \le \mathrm{K}\,[\mathrm{K}\,\| d(\mathrm{T}x_{2n(i)}, \mathrm{T}u) \| + \| d(\mathrm{T}x_{2n+1(i+1)}, \mathrm{T}u) \|] \to\infty \quad (3.1.8)$$

Where K is normal constant of X.
Hence

$$\| d(\mathrm{TR}u, \mathrm{T}u) \| = 0 \quad (3.1.9)$$

This implies $\mathrm{TR}u = \mathrm{T}u$. Since T is one to one $\mathrm{R}u = u$.
So, u is a fixed point of R.
Now, if v is another fixed point of R. then

$$d(\mathrm{T}u, \mathrm{T}v) = d(\mathrm{TR}u, \mathrm{TR}v) \le \mathrm{K}\,((\mathrm{T}u, \mathrm{T}v) \quad (3.1.10)$$

Since P is normal cone with normal constant K.
So from (3.1.10), we get

$$\| d((\mathrm{T}u, \mathrm{T}v) \| = 0 \text{ and } (\mathrm{T}u, = \mathrm{T}v) \quad (3.1.11)$$

Since T is continuous. So v is fixed point of R.

Similarly, it can be established that $\mathrm{TS}u = \mathrm{T}u$. Hence $\mathrm{TR}u = \mathrm{T}u = \mathrm{TS}u$

Thus T $u = u$ is the common fixed point pairs of T, R, S in X.
This completes the proof.

Corollary 3.2
Let (X, d) be a complete cone metric space, P be a normal cone with normal constant K. Assume that T: X →X be a continuous and one to one mapping and suppose the mapping R, S: X →X for some positive integer n,
$d(TR^{2n+1}x, TS^{2n+2}y) \leq K\, d(T\,x, Ty)$, for all x, y ∊ X, where K ∊ [0,1] is a constant. Then T, R and S have a unique common fixed point in X.
Proof:
From Theorem (3.1.1), TR^{2n+1} has a unique fixed point $Tu = u$.

But $TR^{2n+1}(TRu) = TR\,(TR^{2n+1}u) = TRu$, so T Ru is also fixed point of TR^{2n+1}. Hence $TRu = Tu$. T is one to one. Then $Ru = u$ has a fixed point of R. If v is another fixed point of R. And T is one to one, we get $Tu == Tv$.
Since the fixed point of R is also fixed point of TR^{2n+1}, so the fixed point of R is unique.

Similarly, it can be established that TS u = T u.

$$\text{Hence } TRu = u = TSu.$$

Thus u is common fixed point of T, R and S in X.

Theorem 3.3 Let (X, d) be a complete cone metric space, P be a normal cone with normal constant. Assume that T: X →X is a continuous and one to one mapping. Suppose R, S: X →X be a pair maps satisfying the contractive condition.

$$d\,(TRx, T\,Sy) \leq K\,[d\,(T\,Rx, T\,x) + d\,(TSy, Ty)] \tag{3.3.1}$$

For all x, y ∊ X, where K ∊ [0, ½] is a constant. Then R and S have a unique common fixed point in X. And for any x ∊ X, iterative sequences $\{T\,R^{2n+1}x\}$ and $\{T\,S^{2n+2}x\}$ converge to common fixed point.
Proof: Choose x_0∊ X, Set $x_1 = TRx,_0$, $x_3, = TRx_2 = TR^3x_0$---------- $x_{2n+1} = TRx_{2n} = TR^{2n+1}x_0$---
Similarly, we can have $x_2 = TSx_1 = TS^2x_0$, $x_4 = TSx_3 = TS^4x_0$------------ x_{2n+2} $TSx_{2n+1} = TS^{2n+2}x_0$ --------
We have,

$$d\,(Tx_{2n+1}, T\,x_{2n}) = d(TR\,x_{2n}, TSx_{2n-1}) \leq d\,(Tx_{2n}, TSx_{2n}) + d(Tx_{2n-1}, TSx_{2n-1})$$
$$\leq K\,[d\,(Tx_{2n-1}, T\,x_{2n}) + d\,(T\,x_{2n}, Tx_{2n-1})]$$

$$\text{So } d\,(Tx_{2n-1}, Tx_{2n}) \leq \frac{K}{1-K}[d(Tx_{2n}, Tx_{2n-1})]$$

Where $h = \frac{K}{1-K}$

For n > m,

$$d(Tx_{2n}, Tx_{2m}) \le d(Tx_{2n}\, Tx_{2n-1}) + \boldsymbol{x}(Tx_{2n-1}, Tx_{2n--2}) + \text{--------} + d(Tx_{2n-+1}, x_{2n})$$
$$\le (h^{2n-1} + h^{2n-2} + \text{-----} + h^{2m})\, d\,(T\,x_1, T\,x_0)$$

$$\le \frac{h^{2m}}{1-h}\, d\,(Tx_1, Tx_0) \qquad (3.3.3)$$

Since $P \subset E$ normal cone with normal constant, so by (3.3.3) we get

$$\|\, d\,(Tx_{2n}, Tx_{2m})\,\| \le \frac{h^{2m}}{1-h}\, \|\, d\,(Tx_1, Tx_0)\,\|$$

This implies that $d(Tx_{2m}, Tx_{2m}) \to 0$ (n, m→∞).

Hence $\{Tx_{2m}\}$ is a Cauchy Sequence. Since by the completeness of X there exists $v \in X$ such that

$$Tx_{2n} \to v\ (n\to\infty)$$

$$i.e. \quad \lim_{n\to\infty} Tx_{2n} = v \qquad (3.3.4)$$

Now if T is a subsequently convergent, $\{x_{2n}\}$ has a convergent sub sequence. So there exist $u \in X$ and

$\{x_{2n(i)}\}$ such that $\lim_{n\to\infty} x_{2n(i)} = u$ (3.3.5)

Since T is continuous and by (3.3.5), we obtain

$$\lim_{n\to\infty} Tx_{2n\,(i)} = Tu \text{------------------------------} \qquad (3.3.6\,)$$

by (3.3.4) and (3.3.6), we conclude that $T\,u = v$ (3.3.7)

$$\text{Since } d\,(TRu, Tu) \le d(TRx_{2n}, TRu) + d(TRx_{2n}, Tu)$$

$$\le K\,[d\,(Tx_{2n}, Tx_{2n}) + d\,(TRu, Tu)] + d\,(Tx_{2n+1}, Tu)$$

$$d\,(TRu, Tu) \le \frac{1}{1-K}[K\, d\,(TRx_{2n}, Tx_{2n}) + d\,(Tx_{2n+1}, Tu)]$$

Now, again since $P \subset E$ is cone normal with normal constant K.

So we get.

$$\|\, d(TRu, Tu)\,\| \le K.\frac{1}{1-K}[K\ \|\, d\,(Tx_{2n+1}, Tx_{2n})\,\| + \|\, d\,(Tx_{2n+1}, Tu)\,\|] \to 0 \quad (3.3.8)$$

$$\text{Hence}\|\, d\,(TRu, Tu)\,\| = 0 \qquad (3.3.9)$$

This implies that $TRu = Tu$. So u is a fixed point of R. Since T is one to one . Then $Ru = u$ is fixed point of R.

Now if $v \in X$ is another fixed point of R then

$$d\,(TRu, Tu) \le d\,(TRu, TRv)$$
$$\le K\,[d\,(TRu, Tv) + d\,(TRu, Tv)] = 0$$

Hence Tu = Tv

Therefore the unique fixed point of R.

Similarly, it can be established that TSu = Tu. Hence TRu = Tu= TSu.

Thus u is the common fixed point of pair of maps T, R and S.

Hence the pair (T, R) & (T, S) have a unique common fixed point.

Theorem 3.4 Let (X, d) be a complete cone metric space, P be a normal cone with normal constant. Assume that T: X →X be continuous one to one mappings. Suppose that the mapping R, S: X →X be a pair maps satisfy the contractive condition.

$$d\,(T\,Rx, T\,Sy) \le K\,[\,d\,(TRx, Ty) + d\,(TSy, Tx)] \qquad (3.4.1)$$

For all x, y ∊ X, where K ∊ [0, ½] is a constant. Then the pair T, R and S have a unique common fixed point.

Proof: Choose x_0∊ X, set $x_1 = TRx_0$, $x_3 = TRx_2 = TR^3x_0$---------- $x_{2n+1} = TRx_{2n} = TR^{2n+1}x_0$----------

Similarly, we can have $x_2 = TSx_1 = TS^2x_0$, $x_4 = TSx_3 = TS^4x_0$------------$x_{2n+2} = TSx_{2n+2} = TS^{2n+2}x_0$ - -------

We have

$$d(Tx_{2n+1}, Tx_{2n}) = d(TRx_{2n}, TSx_{2n-1})$$

$$\le K\,[d\,(TRx_{2n}, Tx_{2n-1}) + d\,(TSx_{2n-1}, Tx_{2n})]$$

$$\le K\,[d\,(Tx_{2n-1}, Tx_{2n}) + d\,(Tx_{2n}, Tx_{2n-1})]$$

So, $$d\,(Tx_{2n+1}, Tx_{2n}) \le \frac{K}{1-K}\,d\,(Tx_{2n}, Tx_{2n-1}) \qquad (3.4.2)$$

Where $\frac{K}{1-K} = h$

So, now we have

$$d\,(Tx_{2n+1}\,Tx_{2n}) \le h\,d(Tx_{2n}\,Tx_{2n-1}) \qquad (3.4.3)$$

For n > m,

$$d(Tx_{2n}, Tx_{2m}) \le d(Tx_{2n}, Tx_{2n-}) + d(Tx_{2n-1}, Tx_{2n--2}) + \text{-------------} + d(Tx_{2n-1}, Tx_{2m})$$

$$\le (h^{2n-1} + h^{2n-2} + \text{-----} + h^{2m})\,d\,(Tx_1, Tx_0)$$

$$\le \frac{h^{2m}}{1-h}\,d\,(Tx_1, Tx_0) \qquad (3.4.4)$$

Since P be a normal cone with normal constant K, So we get

$$\| d(Tx_{2n}, Tx_{2m}) \| \leq \frac{h^{2m}}{1-h} K \| d(Tx_1, Tx_0) \| \quad (3.4.5)$$

This implies that $d(Tx_{2n}, Tx_{2m}) \to 0$ $(n, m \to \infty)$.

Hence $\{Tx_{2n}\}$ is a Cauchy Sequence. By the completeness of X, there exists $v \in X$ such that

$$Tx_{2n} \to v \ (n \to \infty) \ i.e. \quad \lim_{n\to\infty} Tx_{2n} = v. \quad (3.4.6)$$

If T subsequently convergent, $\{x_{2n}\}$ has a convergent sub sequence. So there exist $u \in X$ and $\{x_{2n(i)}\}$ such that $\lim_{n\to\infty} x_{2n(i)} = u$ (3.4.7)

Since T is continuous and by (3.4.7), we obtain

$$\lim_{n\to\infty} Tx_{2n(i)} = Tu \text{----------------} \quad (3.4.8)$$

by (3.4.6) and (3.4.8), we conclude that $Tu = v$ (3.4.9)

Since

$$d(TRu, Tu) \leq d(TRx_{2n(i)}, TRu) + d(TRx_{2n(i)}, Tu)$$

$$\leq K[d(TRu, Tu) + d(Tx_{2n}, Tu) + d(Tx_{2n+1(i+1)}, Tu)] + d(Tx_{2n+1(i+1)}, Tu)$$

$$d(TRu, Tu) \leq \frac{1}{1-K} [K d(TRx_{2n}, Tx_{2n}) + d(Tx_{2n+1(i+1)}, Tu)] + d(Tx_{2n+1(i+1)}, Tu)$$

Since P is a cone normal with normal constant K.

So, we get

$$\| d(TRu, Tu) \| \leq K.\frac{1}{1-K} [K \| d(Tx_{2n+1}, Tu) \| + \| d(Tx_{2n+1(i+1)}, Tu) \|] + \| d(Tx_{2n+1(i+1)}, Tu) \| \to 0$$

Hence $\| d(TRu, Tu) \| = 0$

This implies that $TRu = Tu$. So u is a fixed point of R. Now if $v \in X$ is another fixed point of R then

$$d(Tu, Tv) \leq d(TRu, TRv)$$

$$\leq K[d(TRu, Tv) + d(TRv, Tu)$$

$$= 2K\, d(Tu, Tv)$$

Hence $d(Tu, Tv) = 0$, $Tu = v$

Therefore $u = v$ is a unique fixed point of R.

Similarly, it can be established that TSu = Tu.

Hence TRu = Tu= TSu. Thus u is the unique common fixed point of pair of maps T, R and S.

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