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On a Class of Weakly Berwald (α, β)- metrics of Scalar flag curvature

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Abstract

In this paper, we study two important class of (α, β)-metrics $F = \alpha + k\beta^2/\alpha$ (where $k \neq 0$ a constant) and $F = \alpha + \beta + \beta^2/\alpha + \beta^3/\alpha^2$ are of scalar flag curvature. We proved that these metrics are weak Berwald if and only if they are Berwald and their flag curvature vanishes. Further, we found these metrics are locally Minkowskian.

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1. Introduction

In Finsler geometry, the flag curvature $\boldsymbol{K}$(P, y) is an analogue of the sectional curvature in Riemann geometry. One of the fundamental problems in Riemann-Finsler geometry is to study and characterize Finsler metrics of scalar flag curvature. It is known that every Berwald metric is a Landsberg metric and also, for any Berwald metric the S-curvature vanishes [3]. In 2003, Z. Shen proved that Randers metrics with vanishing S-curvature and $\boldsymbol{K}$=0 are not Berwaldian [17]. He also proved that the Bishop-Gromov volume comparison holds for Finsler manifolds with vanishing S-curvature [19].

The concept of (α, β)-metrics and its curvature properties have been studied by various authors ([11], [2], [10], [12], [7], [9], [22], [13]). X. Cheng, X. Mo and Z. Shen (2003) have obtained the results on the flag curvature of Finsler metrics of scalar curvature [4]. Yoshikawa, Okubo and M. Matsumoto (2004) showed the conditions for some (α, β)-metrics to be weakly- Berwald [22]. Z. Shen and Yildirim (2008) obtained the necessary and sufficient conditions for the metric $F = (\alpha + \beta)^2/\alpha$ to be projectively flat [20]. They also obtained the necessary and sufficient conditions for the metric $F = (\alpha + \beta)^2/\alpha$ to be projectively flat Finsler metric of constant flag curvature and proved that, in this case, the flag curvature vanishes [20]. Xiang and X. Cheng (2009) obtained the conditions for the metric $F = (\alpha + \beta)^{m+1}/\alpha^m$ to be weakly-Berwald [9]. Recently, X. Cheng(2010) has worked on (α, β)-metrics of scalar flag curvature with constant S-curvature [6].

The main purpose of the present paper is to study and characterize the two important class of weakly-Berwald (α, β)-metrics $F = \alpha + k\beta^2/\alpha$ (where $k \neq 0$ a constant) and $F = \alpha + \beta + \beta^2/\alpha + \beta^3/\alpha^2$ are of scalar flag curvature. The terminologies and notations are referred to [6], [3].

2. Preliminaries

Let M be an n-dimensional C^∞ manifold and $TM = \bigcup_{x \in M} T_x M$ denote the tangent bundle of M. A Finsler metric on M is a functions $F : TM \to [0, \infty)$ with the following properties:

a) F is C^∞ on $TM \backslash \{0\}$;

b) At each point $x \in M$, $F_x(y) = F(x, y)$ is a Minkowskian norm on $T_x M$. The pair *(M, F)* is called Finsler manifold;

Let *(M, F)* be a Finsler manifold and

$$g_{ij}(x, y) = \frac{1}{2}[F^2(x, y)]_{y^i y^j}. \tag{1}$$

For a vector $y = y^i \frac{\partial}{\partial x^i}|_x \neq 0$, F induces an inner product g_y on $T_x M$ as follows

$$g_{ij}(u, v) = g_{ij} u^i v^j,$$

where $u = u^i \frac{\partial}{\partial x^i}|_x$ and $u = u^i \frac{\partial}{\partial x^i}|_x$. Further, the Cartan torsion C and the mean Cartan torsion I are defined as follows [3]:

$$\begin{aligned} C_y(x, y) &= C_{ijk} u^i v^j w^k, \\ I_y(u) &= I_i(x, y) u^i, \end{aligned} \tag{2}$$

where

$$C_{ijk}(x, y) = \frac{1}{4}[F^2] y^i y^j y^k(x, y), \tag{3}$$

$$I_i(x, y) = g^{jk} C_{ijk}(x, y) = \frac{\partial}{\partial y^i}[ln\sqrt{det(g_{jk})}], \tag{4}$$

where $(g^{jk}) = (g_{jk})^{-1}$.

Let $\alpha = \sqrt{a_{ij} y^i y^j}$ be a Riemannian metric and $\beta = b_i(x) y^i$ be a 1-form on an n-dimensional manifold M. The norm

$$\|\beta_x\|_\alpha = \sqrt{a^{ij}(x) b_i(x) b_j(x)},$$

Let $\phi = \phi(s)$ be a C^∞ positive function on an open interval $(-b_0, b_0)$ satisfying the following conditions

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, |s| \le b < b_0.$$

Then the functions $F = \alpha\phi(\beta/\alpha)$ is a Finsler metric if and only if $\|\beta_x\|_\alpha < b_0$. Such Finsler metrics are called (α, β)-metrics.

Let ";" and "|" denote the horizontal covariant derivative with respect to F and α, respectively.

Let

$$r_{ij} = \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i}), \tag{5}$$

$$s^i_j = a^{ik} s_{kj}, \quad s_i = b^j s_{ji} = b_j s^j_i, \quad r_i = b^j r_{ji}. \tag{6}$$

For a C^∞ positive function $\phi = \phi(s)$ on $(-b_0, b_0)$ and a number $b \in [0, b_0)$, let

$$\Phi = -(Q - sQ')\{n\Delta + 1 + sQ\} - (b^2 - s^2)(1 + sQ)Q'', \tag{7}$$

where $\Delta = 1 + sQ + b^2 - s^2 Q'$ and $Q = \phi'/(\phi - s\phi')$.

The geodesic *x=x(t)* of a Finsler metric F is characterized by the following system of second order ordinary differential equations:

$$\frac{d^2 x^i(t)}{dt^2} + 2G^i(x(t), x'(t)) = 0,$$

where

$$G^i = \frac{1}{4} g^{ij}\{[F^2]_{x^m y^j} y^m - [F^2]_{x^j}\},$$

G^i is called as geodesic coefficients of F.

For an (α, β)-metric $F = \alpha\phi(\beta/\alpha)$, $s = \beta/\alpha$, using a Maple program, we can get the following [3]:

$$G^i = \tilde{G}^i + \alpha Q s^i_0 + \Theta\{-2\alpha Q s_0 + r_{00}\}\{\frac{y^i}{\alpha} + \frac{Q'}{Q - sQ'} b^i\}, \tag{8}$$

where $\tilde{G}^i$ denote the spray coefficients of α.

We shall denote $r_{00} = r_{ij}y^i y^j, s_{i0} = s_{ij}y^j, s_0 = s_i y^i$ and $\Theta = \dfrac{Q - sQ^{'}}{2\Delta}$.

Furthermore, let

$$\begin{aligned} h_i &= \alpha b_i - sy_i, \\ \Psi_1 &= \sqrt{b^2 - s^2} \\ &\quad \Delta^{1/2}[\frac{\sqrt{b^2 - s^2}}{\Delta^{3/2}}\Phi]^{'}, \\ \Psi_2 &= 2(n+1)(Q - sQ^{'}) + 3\frac{\Phi}{\Delta}. \end{aligned}$$

By a direct computation, we can obtain a formula for the mean Cartan torsion of (α, β)-metrics as follows [8]:

$$I_i = -\frac{\Phi(\phi - s\phi^{'})}{2\Delta\phi\alpha^2}(\alpha b_i - sy_i). \tag{9}$$

According to Diecke's theorem [16], a Finsler metric is Riemannian if and only if the mean Cartan torsion vanishes, I=0. Clearly, an (α, β)-metric $F = \alpha\phi(\beta/\alpha), s = \beta/\alpha$ is Riemannian if and only if $\Phi = 0$(see [8]).

For a Finsler metric *F=F(x, y)* on a manifold *M*, the Riemann curvature $\mathbf{R}_y = R^i_k \frac{\partial}{\partial x^i} \otimes dx^k$ is defined by

$$R^i_k = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^m \partial y^k}y^m + 2\frac{\partial^2 G^i}{\partial y^m \partial y^k} - \frac{\partial G^i}{\partial y^m}\frac{\partial G^m}{\partial y^k}.$$

Let $R_{jk} = g_{ji}R^i_k$, then $R_{jk}y^j = 0$, $R_{jk} = R_{kj}$.

For a flag {P, y}, where P= span{y, u}$\subset T_xM$, the flag curvature ***K=K***(P, y)of *F*is defined by

$$\boldsymbol{K}(P, y) = \frac{R_{jk}(x, y)u^j u^k}{F^2(x, y)h_{jk}(x, y)u^j u^k}, \tag{10}$$

where $h_{jk} = g_{jk} - F^{-2}g_{jp}y^p g_{kq}y^q$.

We say that Finlser metric F is of scalar flag curvature if the flag curvature ***K=K***(x, y) is independent of the flag P. By the definition, *F* is of scalar flag curvature ***K=K***(x, y) if and only if in a standard local coordinate system,

$$R^i_k = \boldsymbol{K}F^2 h^i_k, \tag{11}$$

where $h^i_k = g^{ij}h_{jk} = g^{ij}FF_{y^j y^k}$([3][16]).

The Schur Lemma [16] in Finsler geometry tell us that, in dimension $n \geq 3$, if F is of isotropic flag curvature, $\boldsymbol{K}$=$\boldsymbol{K}$(x), then it is of constant flag curvature, $\boldsymbol{K}$= constant.

The Berwald curvature $\boldsymbol{B}_y = B^i_{jkl} dx^j \otimes \frac{\partial}{\partial x^i} \otimes dx^k \otimes dx^l$ and mean Berwald curvature $\boldsymbol{E}_y = E_{ij} dx^i \otimes dx^j$ are defined respectively by

$$B^i_{jkl} = \frac{\partial^3 G^i(x,y)}{\partial y^j \partial y^k \partial y^l}, E_{ij} = \frac{1}{2} \frac{\partial^2}{\partial y^i \partial y^j} (\frac{\partial G^m}{\partial y^m}) = \frac{1}{2} B^m_{ijm}.$$

A Finsler metric F is called weak Berwald metric if the mean Berwald curvature vanishes, i.e., (**E**=0)**B**=0. A Finsler metric F is said to be of isotropic mean Berwald curvature if **E**$=\frac{1}{2}(n+1)c(x)F^{-1}h$, where c=$c(x)$ is a scalar function on the manifold M.

Theorem 2.1. [17] For special (α, β)-metric $F = \alpha + k\beta^2/\alpha$, where $k \neq 0$ is constant and Matsumoto metric (2^{nd}appx.) $F = \alpha + \beta + \beta^2/\alpha + \beta^3/\alpha^2$on an n-dimensional manifold M. Then the following are equivalent:

(a) F is of isotropic S-curvature, $S = (n+1)c(x)F$;

(b) F is of isotropic mean Berwald curvature, $E = \frac{n+1}{2} c(x)F^{-1}h$;

(c) β is a killing 1-form with constant length with respect to α, i.e., $r_{00} = 0$ and $s_0 = 0$;

(d) S-curvature vanishes, $\boldsymbol{S}$=0;

(e) F is a weak Berwald metric, $\boldsymbol{E}$=0; where c=c(x) is scalar function on the manifold M.

Note that, the discussion in [14] doesn't involve whether or not F is Berwald metric. By the definitions, Berwald metrics must be weak Berwald metrics but the converse is not true. For this observation, we further study the metrics $F = \alpha + k\beta^2/\alpha$ ($k \neq 0$ is constant) and $F = \alpha + \beta + \beta^2/\alpha + \beta^3/\alpha^2$.

For a Finsler metric F=$F(x, y)$ on and n-dimensional manifold M, the Busemann-Hausdorff volume form $dV_F := \sigma_F(x) dx^1 \wedge \cdots \wedge dx^n$ is given by

$$\sigma_F(x) = \frac{Vol(B^n(1))}{Vol\{(y^i) \in R^n | F(x,y) < 1\}},$$

Vol denotes the Euclidean volume in R^n. The S-curvature is given by

$$S(x,y) = \frac{\partial G^m}{\partial y^m} - y^m \frac{\partial (ln\sigma_F)}{\partial x^m}, \tag{12}$$

Clearly, the mean Berwald curvature $\boldsymbol{E}_y$=$E_{ij}dx^i \otimes dx^j$ can be characterized by use of S-curvature as follows:

$$E_{ij} = \frac{1}{2}\frac{\partial^2 S}{\partial y^i \partial y^j}.$$

A Finsler metric $F$is said to be of isotropic S-curvature if $\boldsymbol{S}$=$(n+1)c(x)F(x,y)$, where *c=c(x)* is a scalar function on the manifold M. S-curvature is closely related to the flag curvature. We use the following important result proved by [4]:

Theorem 2.2. Let (M, F) be an n-dimensional Finsler manifold of scalar flag curvature with flag curvature $\boldsymbol{K}$=$\boldsymbol{K}$(x, y). Suppose that the S-curvature is isotropic, $\boldsymbol{S}$=$(n+1)c(x)F(x,y)$, where *c=c(x)* is a scalar function on M. Then there is a scalar function $\sigma(x)$ on M such that

$$\boldsymbol{K} = \frac{3c_{x^m}(x)y^m}{F(x,y)} + \sigma(x). \tag{13}$$

This shows that S-curvature has important influence on the geometric structures of Finsler metrics. For a Finsler metric F, the Landsberg curvature $\boldsymbol{L} = L_{ijk}dx^i \otimes dx^j \otimes dx^k$ and the mean Landsberg curvature $\boldsymbol{J}$= $J_k dx^k$ are defined respectively by

$$L_{ijk} = -\frac{1}{2}FF_{y^m}[G^m]_{y^i y^j y^k}, \quad J_k = g^{ij}L_{ijk}.$$

A Finsler metric F is called weak Landsberg metric if the mean Landsberg curvature vanishes, i.e., ($\boldsymbol{J}$=0) $\boldsymbol{L}$=0. For an (α,β)-metric $F=\alpha\phi(s)$, $s=\beta/\alpha$, Li Benling and Z. Shen [10] obtained the following formula of the mean Landsberg curvature

$$\begin{aligned} J_i = -\frac{1}{2\Delta\alpha^4}\Bigg\{ & \frac{2\alpha^2}{b^2-s^2}\left[\frac{\Phi}{\Delta} + (n+1)(Q - sQ^{'})\right](s_0+r_0)h_i \\ & + \frac{\alpha}{b^2-s^2}\left[\Psi_1 + s\frac{\Phi}{\Delta}\right](r_{00} - 2Q\alpha s_0)h_i \\ & + \alpha[-\alpha Q^{'}s_0h_i + \alpha Q(\alpha^2 s_i - y_i s_0)] + \alpha^2\Delta s_{i0} \\ & + [\alpha^2(r_{i0} - 2\alpha Q s_i) - (r_{00} - 2\alpha Q s_0)y_i]\frac{\Phi}{\Delta}\Bigg\}. \end{aligned} \tag{14}$$

Besides, they also obtained

$$\bar{J} = J_i b^i = -\frac{1}{2\Delta\alpha^2}\{\Psi_1(r_{00} - 2\alpha Q s_0) + \alpha\Psi_2(r_0+s_0)\}. \tag{15}$$

The horizontal covariant derivatives $J_{i;m}$ and $J_{i|m}$ of J_i with respect to F and α respectively are given by

$$J_{i;m} = \frac{\partial J_i}{\partial x^m} - J_l\Gamma^l_{im} - \frac{\partial J_i}{\partial y^l}N^l_m,$$

$$J_{i|m} = \frac{\partial J_i}{\partial x^m} - J_l\bar{\Gamma}^l_{im} - \frac{\partial J_i}{\partial y^l}\bar{N}^l_m,$$

where $\Gamma^l_{ij} = \dfrac{\partial G^l}{\partial y^i \partial y^j}$, $N^l_j = \dfrac{\partial G^l}{\partial y^j}$ and $\bar{\Gamma}^l_{ij} = \dfrac{\partial \bar{G}^i}{\partial y^i \partial y^j}$, $\bar{N}^l_j = \dfrac{\partial \bar{G}^l}{\partial y^j}$.

Further we have,

$$\begin{aligned} J_{i;m} y^m &= \{J_{i|m} - J_l(\Gamma^l_{im} - \bar{\Gamma}^l_{im}) - \frac{\partial J_i}{\partial y^l}(N^l_m - \bar{N}^l_m)\} y^m \\ &= J_{i|m} y^m - J_l(N^l_i - \bar{N}^l_i) - 2\frac{\partial J_i}{\partial y^l}(G^l - \bar{G}^l). \end{aligned}$$

If a Finsler metric F is of constant flag curvature $\boldsymbol{K}$(see[16]), then

$$J_{i;m} y^m + \boldsymbol{K} F^2 I_i = 0.$$

So, if an (α,β)-metric $F = \alpha\phi(s)$, $s = \beta/\alpha$, is of constant flag curvature $\boldsymbol{K}$, then

$$J_{i|m} y^m - J_l \frac{\partial (G^l - \bar{G}^l)}{\partial y^i} - 2\frac{\partial J_i}{\partial y^i}(G^l - \bar{G}^l) + \boldsymbol{K}\alpha^2\phi^2 I_i = 0.$$

Contracting the above equation by b^i yields the following equation

$$\bar{J}_{|m} y^m - J_i a^{ik} b_{k|m} y^m - J_l \frac{\partial (G^l - \bar{G}^l)}{\partial y^i} b^i - 2\frac{\partial \bar{J}}{\partial y^l}(G^l - \bar{G}^l) + \boldsymbol{K}\alpha^2\phi^2 I_i b^i = 0. \quad (16)$$

3. Characterization of Weakly-Berwald (α,β)-metrics of Scalar flag curvature

In n-dimensional Finsler manifold ($n \geq 3$), we characterize the two important class of Weakly-Berwald (α,β)-metrics of Scalar flag curvature. So first we have to prove the following lemma.

Lemma 3.1. Let two (α,β)-metrics, $F = \alpha + k\beta^2/\alpha$ (where $k \neq 0$ a constant) and $F = \alpha + \beta + \beta^2/\alpha + \beta^3/\alpha^2$ are of non-Randers type if $\Phi \neq 0$.

Proof. ***Case-I:*** $F = \alpha + k\beta^2/\alpha$ (where $k \neq 0$ a constant).
In this case, by a direct computation(7), we have

$$\Phi = -\frac{A\phi}{(1 - ks^2)^4},$$

where

$$\begin{aligned} \phi &= 1 + ks^2, \\ A &= (4k^2 + 4k^3(3n+1))s^5 - 16k^3 s^4 - (4k^2(n+1) \\ &\quad + 8k^3 n b^2 + 4k^2 b^2)s^3 + 4(4k^3 b^2 - k^2)s^2 + 4k^2 b^2. \end{aligned}$$

Assume that $\Phi = 0$. Then A=0. Multiplying A=0 with α^5 yields

$$[4k^2\beta^5(1 + 12k^3n + 4k^3) - 4k^2((n + 1) + 2knb^2 + b^2)\beta^3\alpha^2] \\ + \alpha[4k^2b^2\alpha^4 + 4k^2(4kb^2 - 1)\beta^2\alpha^2 - 16k^3\beta^4] = 0.$$

Hence we have,

$$4k^2\beta^5(1 + 12k^3n + 4k^3) - 4k^2((n + 1) + (2kn + 1)b^2)\beta^3\alpha^2 = 0, \tag{17}$$

$4k^2b^2\alpha^4 + 4k^2(4kb^2 - 1)\beta^2\alpha^2 - 16k^3\beta^4 = 0.$

Clearly, observe that β^5 is not divisible by α^2. Since k=0 by (17), which is a contradiction with $k \neq 0$.

Case-II: Similarly, for the Matsumuto 2nd approximation metric, $F = \alpha + \beta + \beta^2/\alpha + \beta^3/\alpha^2$, we have

$$\Phi = -\frac{A\phi}{(1 - s^2 - 2s^3)^5},$$

where,

$$\phi = 1 + s + s^2 + s^3,$$
$$A = A_0 + A_1s - A_2s^2 + A_3s^3 + A_4s^4 + A_5s^5 + A_6s^6 + A_7s^7 + A_8s^8 \\ + A_9s^9 - A_{10}s^{10} - A_{11}s^{11},$$

where,

$$\begin{aligned}
A_0 &= [b^2(2n + 1) + (n + 1)],\\
A_1 &= 3b^2(2n + 1),\\
A_2 &= 2[b^2(7n + 4) + (5n + 1)],\\
A_3 &= -[10b^2(7n - 1) + (22n + 13)],\\
A_4 &= [-b^2(98n + 6) + (14n - 13)],\\
A_5 &= 3b^2(20n - 9) + 4(27n + 5),\\
A_6 &= 2b^2(145n - 18) + 2(85n + 33),\\
A_7 &= 2b^2(110n - 12) + (58n + 13),\\
A_8 &= -3b^2(8n + 9) - (183n + 73),\\
A_9 &= -2[(255n + 78) + 3b^2(24n + 1)],\\
A_{10} &= 6(96n + 17),\\
A_{11} &= 6(32n + 9).
\end{aligned}$$

Again assume that $\Phi = 0$ and A=0, then

$$0 = A_0 + A_1s - A_2s^2 + A_3s^3 + A_4s^4 + A_5s^5 + A_6s^6 + A_7s^7 \\ + A_8s^8 + A_9s^9 - A_{10}s^{10} - A_{11}s^{11}.$$

Hence,

$$A_1\beta\alpha^{10} + A_3\beta^3\alpha^8 + A_5\beta^5\alpha^6 + A_7\beta^7\alpha^4 + A_9\beta^9\alpha^2 - A_{11}\beta^{11} = 0, \tag{18}$$

$$A_0\alpha^{10} - A_2\beta^2\alpha^8 + A_4\beta^4\alpha^6 + A_6\beta^6\alpha^4 + A_8\beta^8\alpha^2 - A_{10}\beta^{10} = 0.$$

Note that β^{11} is not divisible by α^2. As in previous case here also $k = 0$ by (18), which is a contradiction with $k \neq 0$.

From above two cases, it is quiet clear that $\Phi \neq 0$ (that is, (α, β)-metrics $F = \alpha + k\beta^2/\alpha$(where $k \neq 0$ a constant) and $F = \alpha + \beta + \beta^2/\alpha + \beta^3/\alpha^2$ are non-Riemannian). ■

By using this lemma 3.1, now we can prove the following

Theorem 3.2. Let $F = \alpha + k\beta^2/\alpha$ ($k \neq 0$,a constant) be a Finsler space with (α, β)-metric is of scalar flag curvature ***K***=***K*** (x, y). Then F is weak Berwald metric if and only if F is Berwald metric and ***K***=0. In this case, F must be locally Minkowskian.

Proof. By the above lemma and (9) we know that (α, β)-metrics $F = \alpha + k\beta^2/\alpha$ can't represents the Riemannian metric, where $k \neq 0$ a constant and $\beta \neq 0$.

The sufficiency is obvious. We just prove the necessity.

First, we assume that the metric F is weak Berwald. By lemma 3.1, we know that $S = (n + 1)c(x)F$ with *c(x)=0* and

$$r_{00} = 0, s_0 = 0. \tag{19}$$

By theorem 2.2 and 3.2, F must be of isotropic flag curvature $\boldsymbol{K}=\sigma(x)$.

Further, by schur lemma [16] F must be of constant flag curvature. From (19), we can simplify (8), (14) and (15) as follows

$$G^i - \bar{G}^i = \alpha Q s^i_0, \quad J_i = -\frac{\Phi s_{i0}}{2\alpha\Delta}, \quad \bar{J} = 0.$$

In addition, from (9), we obtain

$$I_i b^i := -\frac{\Phi(\phi - s\phi^{'})}{2\Delta F}(b^2 - s^2).$$

Thus (16) can be expressed as follows

$$\frac{\Phi s_{i0}}{2\Delta\alpha} a^{ik} s_{k0} + \frac{\Phi s_{l0}}{2\Delta\alpha}(sQs^l_0 + Q^{'} s^l_0(b^2 - s^2)) - \boldsymbol{K} F \frac{\Phi}{2\Delta}(\phi - s\phi^{'})(b^2 - s^2) = 0.$$

By lemma 3.1, we've

$$s_{i0}s^i_0 + s_{l0}(\alpha Q s^l_0)_{.i} b^i - \boldsymbol{K} F\alpha(\phi - s\phi^{'})(b^2 - s^2) = 0.$$

Note that $F = \alpha\phi(s)$, $s = \beta/\alpha$. We have

$$s_{i0}s_0^i\Delta - \boldsymbol{K}\alpha^2\phi(\phi - s\phi^{'})(b^2 - s^2) = 0. \tag{20}$$

For (α, β)-metric $F = \alpha + k\beta^2/\alpha$ ($k\neq 0$,a constant),

$$\Delta = \frac{\phi(1 + 2kb^2 - 3ks^2)}{(1 - ks^2)^2}.$$

Then (20) becomes

$$(1 + 2kb^2 - 3ks^2)s_{i0}s_0^i - \boldsymbol{K}\alpha^2(b^2 - s^2)(1 - ks^2)^3 = 0.$$

Multiplying this equation with α^6 yields

$$-\boldsymbol{K}b^2\alpha^8 + \{k\beta^2(1 + 3kb^2) + (1 + 2kb^2)s_{i0}s_0^i\}\alpha^6 - 3k\beta^2\{k\beta^2(1 + 2kb^2) + s_{i0}s_0^i\}\alpha^4 + \boldsymbol{K}\beta^6k^2(3 + kb^2)\alpha^2 = \boldsymbol{K}\beta^8k^3. \tag{21}$$

Note that, the left of (21) is divisible by α^2. Hence we can obtain that the flag curvature $\boldsymbol{K}$=0, because $k \neq 0$ and β^8 is not divisible by α^2. Substituting $\boldsymbol{K}$= 0 into (20), we've $s_{i0}s_0^i = a_{ij}(x)s_0^js_0^i = 0$. Because $(a_{ij}(x))$ is positive definite, we've $s_0^i = 0$, i.e., β is closed. By (19), We know that β is parallel with respect to α. Then $F = \alpha + k\beta^2/\alpha$ is a Berwald metric, where $k \neq 0$ a constant. Hence F must be locally Minkowskian. ■

Theorem 3.3. Let $F = \alpha + \beta + \beta^2/\alpha + \beta^3/\alpha^2$ be a Finsler space with (α, β)-metric is of scalar flag curvature $\boldsymbol{K}$=$\boldsymbol{K}$ (x, y). Then F is weak Berwald metric if and only if F is Berwald metric and $\boldsymbol{K}$=0. In this case, F must be locally Minkowskian.

Proof. Consider $F = \alpha + \beta + \beta^2/\alpha + \beta^3/\alpha^2$ is an (α, β)-metric of scalar flag curvature. In this case,

$$\Delta = \frac{\phi(1 - 3s^2 - 8s^3 + 6sb^2 + 2b^2)}{(1 - s^2 - 2s^3)^2}.$$

By lemma 3.1, it is known that F also can't represents the Riemannian metric. The sufficiency is obvious and just only to prove neccessary condition as in the above theorem 3.2. Then (20) becomes

$$(1 - 3s^2 - 8s^3 + 6sb^2 + 2b^2)s_{i0}s_0^i - \boldsymbol{K}\alpha^2(b^2 - s^2)(1 - s^2 - 2s^3)^3 = 0.$$

Implies that $A + \alpha B = 0$, where

$$\begin{aligned} A &= \boldsymbol{K}b^2\alpha^8 + \boldsymbol{K}(3b^2 + 1)\beta^2\alpha^6 + 2\beta[\boldsymbol{K}\beta^5b^2 + 3b^2s_{i0}s_0^i]\alpha^2 \\ &\quad - 3\boldsymbol{K}\beta^4\alpha^4 - 6\beta^3s_{i0}s_0^i - 8\boldsymbol{K}\beta^8, \\ B &= -\boldsymbol{K}\beta^3(b^2\beta^2 - 6)\alpha^4 - \boldsymbol{K}\beta^5(\beta^2 + 6)\alpha^2 \\ &\quad + (1 + 2b^2)s_{i0}s_0^i\alpha^2 - 3\beta^2s_{i0}s_0^i. \end{aligned}$$

Obviously, we have A=0 and B=0. ■

By A=0 and clearly note that β^3 is not divisible by α^2. Then we obtain $s_{i0}s^i_0 = 0$. Hence β is closed. By (19), we know that β is parallel with respect to α. Then F is a Berwald metric. From (20), we find that ***K***=0. Hence F is locally Minkowskian.

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