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Partially Balanced Incomplete Block Designs Arising from Minimum Total Dominating Sets in a Graphs

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Abstract

In this paper we determine the number of minimum total dominating sets of paths and cycles and prove that the set of all minimum total dominating sets of a cycle forms a partially balanced incomplete block design. We also determine all cubic graphs on ten vertices in which the set of all minimum total dominating sets forms a Partially Balanced Incomplete Block Design.

Keywords: PBIBD, Total domination, paths and cycles.

AMS Mathematics Subject Classification: 05C.

Introduction

The relation between Graph theory and partially balanced incomplete block designs (PBIBD) is not a new one and R. C. Bose, in his pioneering paper [1], established the relation between PBIBDs and strongly regular graphs. R. C. Bose[2], has shown that strongly regular graphs emerge from PBIBD; with 2 - association schemes. Harary *et.al.*, [3][4], considered the relation between isomorphic factorization of regular graphs and PBIBD with 2 - association scheme. Ionin and M.S. Shrikhande [5] studied certain kind of designs called (v, k, λ , μ) - designs over strongly regular graphs. Walikar *et.al.,* introduced another kind of design called (v, β_0, μ) - designs whose blocks are maximum independent sets in regular graphs on v vertices.

In this paper, we establish the link between PBIBD and graphs through the collection of minimum total dominating sets. We prove that set of all minimum total dominating sets of cycle forms a PBIBD. We also determine all cubic graphs on ten vertices in which the set of all minimum total dominating sets forms a PBIBD.

Throughout this paper, G = (V, E) stands for a finite, connected, undirected graph with neither loops nor multiple edges. Terms not defined here are used in the sense of Harary [6]. C_n and P_n are cycle and path on n vertices respectively. A set D is a total dominating set if for every vertex v Є V there exists vertex u Є D, u ≠ v such that u is adjacent to v or A subset D of V is called total dominating set in G, if the induced subgraph $\langle D \rangle$ has no isolated vertices. The minimum cardinality of a total dominating set of G is a total domination number of G denoted by γ_t (G) or γ_t . A total dominating set of cardinality γ_t is called minimum total dominating set or γ_t - set. The fundamentals of total domination in graphs and several advanced topics in domination are given in Haynes *et.al.,* [7] [8].

Definitions and Preliminary Results

Definition

[2] Given a set {1, 2, 3, ...,v}, a relation satisfying the following conditions is said to be an association scheme with m- classes.

(i) Any two symbols α, β are i^{th} associates for some i, with $1 \leq i \leq m$ and this relation of being i^{th} associates is symmetric.
(ii) The number of i^{th} associates of each symbol is n_i.
(iii) If α and β are two symbols which are i^{th} associates, then the number of symbols which are j^{th} associates of α and k^{th} associates of β is p^i_{jk}'s and is independent of the pair of i^{th} associates α and β.

Definition

[2] Consider the set of symbols V = {1, 2, 3…,v} and association scheme with m classes. A partially balanced incomplete block design (PBIBD) is a collection of b subsets of S, each of cardinality k (k < v), such that every symbol occurs in exactly r subsets and two symbols α, β which are i^{th} associates occur together in λ_i sets, the number λ_i being independent of choice of pair α, β. The numbers v, b, r, k, λ_i (i = 1, 2, 3, …, m) are called the parameters of the first kind and the numbers n_i's and p^i_{jk}'s of first definition are called the parameters of second kind.

Theorem

[9] For any integer n ≥ 1, $\gamma_t(P_n) = \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{4} \right\rceil - \left\lfloor \frac{n}{4} \right\rfloor$

Theorem

[9] For any integer n ≥ 1, $\gamma_t(C_n) = \left\lceil \frac{n}{\Delta\,(G)} \right\rceil$

Minimum Total Dominating Sets in Paths

Let $M^0_{\gamma_t}(G)$ denote the number of minimum total dominating sets in G. In this section we find $\gamma_t(P_{n)}$ for n = 4k, 4k + 1, 4k + 2, 4k + 3 for any k ≥ 7. Where γ_t is total domination number.

Theorem

$$M^0_{\gamma_t}(P_n) = \begin{cases} 1 & \text{if } n = 4k,\ k \geq 1 \\ 2 & \text{if } n = 4k+1,\ k \geq 7 \\ 5 & \text{if } n = 4k+2,\ k \geq 7 \\ 2 & \text{if } n = 4k+3,\ k \geq 7 \end{cases}$$

Proof: Let $_n = (v_1, v_2, v_3, \dots, v_n)$be the path on n – vertices

Case 1: n = 4k, k ≥ 1

Then $\gamma_t(P_{n)} = 2k$ and $\{v_i, v_{i+1} \,/\, 1 \equiv 2\, mod(4)\}$, 1 ≤ i ≤ 4k is a unique γ_t-set of P_n

There fore $M^0_{\gamma_t}(G) = 1$

Case 2: n = 4k+1, k ≥ 7.

Then $\gamma_t(P_{n)} = 2k + 1$

There exists exactly two γ_t sets of P_n containing $v_1$and v_{4k}

Clearly D_1= $\{v_i, v_{i+1}\, \&\, v_{4k} \,/\, i \equiv 2\, mod(4)\}$

And D_2= $\{v_2, v_i\,, v_{i+1} / \, i \equiv 3\, mod(4)\}$, 1 ≤ i ≤ 4k are the γ_t- sets of P_n

To prove these are the only two sets

Consider $D = D_1 - \{v_2, v_{4k}\}$ or $D = D_2 - \{v_2, v_{4k}\}$ is not a γ_t - set of P_n

Here v_3 and v_{4k} become isolate

Which contradicts the definition of total dominating set.

Thus there are only two γ_t- sets for P_{4k+1}

There fore $M^0_{\gamma_t}(P_n) = 2$

Case 3: n = 4k+2, k ≥ 7

Then $\gamma_t(P_{n)} = 2k + 2$

There exists exactly five γ_t- sets of P_n , which are as follows,

$D_1 = \{v_1, v_2, v_4, v_5, \dots, v_{4k}, v_{4k+1}\}$,

$D_2 = \{v_2, v_3, v_6, v_7, \dots, v_{4k+1}, v_{4k+2}\}$,

$D_3 = \{v_2, v_3, v_6, v_7, \dots, v_{4k}, v_{4k+1}\}$,

$D_4 = \{v_2, v_3, v_4, v_7, v_8, v_{11}, v_{12}, \dots, v_{4k}, v_{4k+1}\}$,

$D_5 = \{v_2, v_3, v_4, v_5, v_8, v_9, v_{12}, v_{13}, \dots, v_{4k}, v_{4k+1}\}$

There fore $M^0_{\gamma_t}(P_n) = 5$

Case 4: n = 4k +3, then $\gamma_t(P_{n)} = 2k + 2$

There exists exactly two γ_t – sets containing v_2 , v_{4k+3} and $v_{1,} v_{4k+2}$

That is D_i= $\{v_j, v_{j+1} \,/\, j \equiv i\, mod(4)\}$, 1 ≤ i ≤ 2 and 1 ≤ j ≤ 4k+2 are the γ_t- set of P_n

There fore $M^0_{\gamma_t}(P_n) = 2$

Minimum Total Dominating Sets in Cycles

Let $M^0_{\gamma_t}(G)$ denote the number of minimum total dominating sets in G and $\gamma_t(G)$ is total domination number.

In this section we find the values of $\gamma_t(C_n)$ for n = 4k, 4k+1, 4k+2, 4k+3.

We have $\gamma_t(C_n) = \left\lceil \frac{n}{\Delta(G)} \right\rceil$, where n = 4k

We proceed to determine $M^0_{\gamma_t}(C_n)$ for cycles.

Theorem

$$M^0_{\gamma_t}(C_n) = \begin{cases} 4 & \text{if } n = 4k,\ k \geq 2 \\ 4k+1 & \text{if } n = 4k+1,\ k \geq 2 \\ 4k+2 & \text{if } n = 4k+2,\ k \geq 2 \\ 4k+3 & \text{if } n = 4k+3,\ k \geq 2 \end{cases}$$

Proof . Let $C_n = \{v_1, v_2, v_3, \ldots, v_{n-1}, v_n, v_1$ be a cycle on n vertices

Case 1: n = 4k, k ≥ 2

$\gamma_t(C_{4k}) = \left\lceil \frac{4k}{2} \right\rceil$ = 2k

Then D_i= $\{v_j, v_{j+1}\ /\ j \equiv i\ mod(4)\}$, 1 ≤ i ≤ 4k, j = 1, 2, 3,…, 4k generates γ_t-sets.

There fore $M^0_{\gamma_t}(C_{4k}) = 4$

Case 2: n = 4k+1, $\gamma_t(C_{4k+1}) = 2k + 1$, k ≥ 2

D_i= $\{v_i, v_{j+1}, v_{j+i+1} / \ j \equiv 5\ mod(4)\}$, 1 ≤ i ≤ 4k + 1, j = 1, 2, 3,…, 4k+1 generates γ_t- sets.

fore $M^0_{\gamma_t}(C_{4k}) = 4k + 1$

Case 3: n = 4k+2, $\gamma_t(C_{4k+2}) = 2k + 2$, k ≥ 2

D_i= $\{v_j, v_{j+1}\ /\ j \equiv i\ mod(4)\}$, 1 ≤ i ≤ 4k + 2 generates γ_t- sets.

There fore $M^0_{\gamma_t}(C_{4k+2}) = 4k + 2$

Case 4: n = 4k+3, $\gamma_t(C_{4k+3}) = 2k + 2$, k ≥ 2

D_i= $\{v_j, v_{j+1}\ /\ j \equiv i\ mod(4)\}$, 1 ≤ i ≤ 4k + 3 , j = i.…4k+3 generates γ_t- sets.

There fore $M^0_{\gamma_t}(C_{4k+3}) = 4k + 3$

Minimum Total Dominating Sets & PBIBDs

We now proceed to establish a relation between the set of all minimum total dominating sets and PBIBDs for cycles and some cubic graphs on ten vertices.

Definition

A graph G is called PBIB graph if the set of all minimum total dominating sets of G forms a PBIBD with a suitable m - association scheme.

Theorem

The collection of all minimum total dominating sets of a cycle C_n , where n = 4k, k ≥ 2 are the blocks of PBIBD with 3 - association scheme and parameters v = n, b = 4, r = 2, k = 4 λ_i , i = 1, 2, 3.

Proof. Let $C_n = \{v_1, v_2, v_3, \dots, v_n, v_1\}$

By the theorem 4.1 $M^0_{\gamma_t}(C_{4k}) = 4$ and four γ_t- sets are given by

$D_i = \{v_j, v_{j+1} \,/\, j \equiv i \, mod(4)\}$, $1 \le i \le 4k$, $j = 1, 2, 3, \dots, 4k$.

Two distinct vertices u and v are said to be first associates if d(u, v) ≡ 0 mod (4); second associates if d(u, v) ≡ 0 mod (2), third associates if d(u, v) = 0.

Clearly the parameters of second kind are given by

$n_1 = \frac{n}{4}$, $n_2 = \frac{n}{2}$ and $n_3 = \frac{n}{4} - 1$ and

$$P^1 = \begin{pmatrix} 0 & 0 & \frac{n}{4} - 1 \\ 0 & \frac{n}{2} & 0 \\ \frac{n}{4} - 1 & 0 & 0 \end{pmatrix}, \quad P^2 = \begin{pmatrix} 0 & \frac{n}{4} & 0 \\ \frac{n}{4} & 0 & \frac{n}{4} - 1 \\ 0 & \frac{n}{4} - 1 & 0 \end{pmatrix}, \quad P^3 = \begin{pmatrix} \frac{n}{4} & 0 & 0 \\ 0 & \frac{n}{2} & 0 \\ 0 & 0 & \frac{n}{4} - 2 \end{pmatrix}$$

The four γ_t - sets D_1, D_2, D_3 *and* D_4 are the blocks of PBIBD with parameters v = n, b = 4, r = 2, k = 4, $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 2$.

Remark

If G is a PBIB graph then the number of γ_t – sets containing any particular vertex v, is the same for all vertices in G.

We now proceed to determine all the cubic graphs with ten vertices in which the set of all minimum total dominating sets forms a PBIBD. We observe that if G is a cubic graph on ten vertices then $\gamma_t(G) = 4$. There are 21 cubic graphs on ten vertices which are given below.

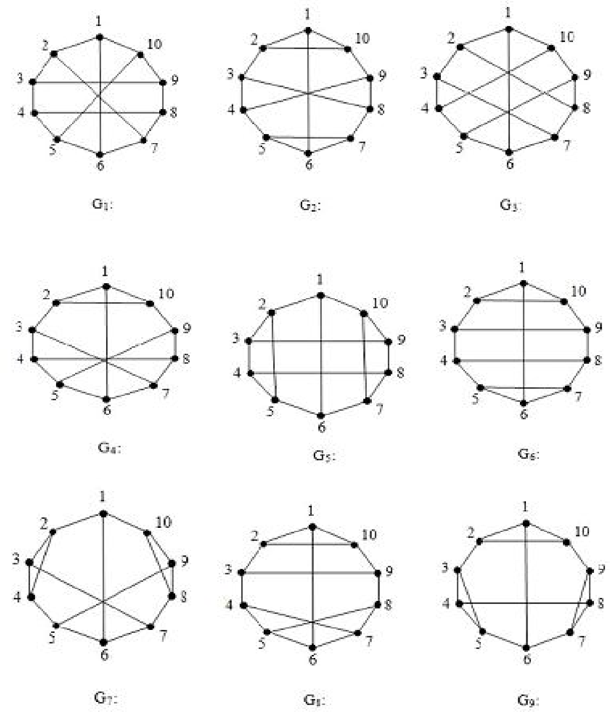

Figure 1: G_1 to G_9.

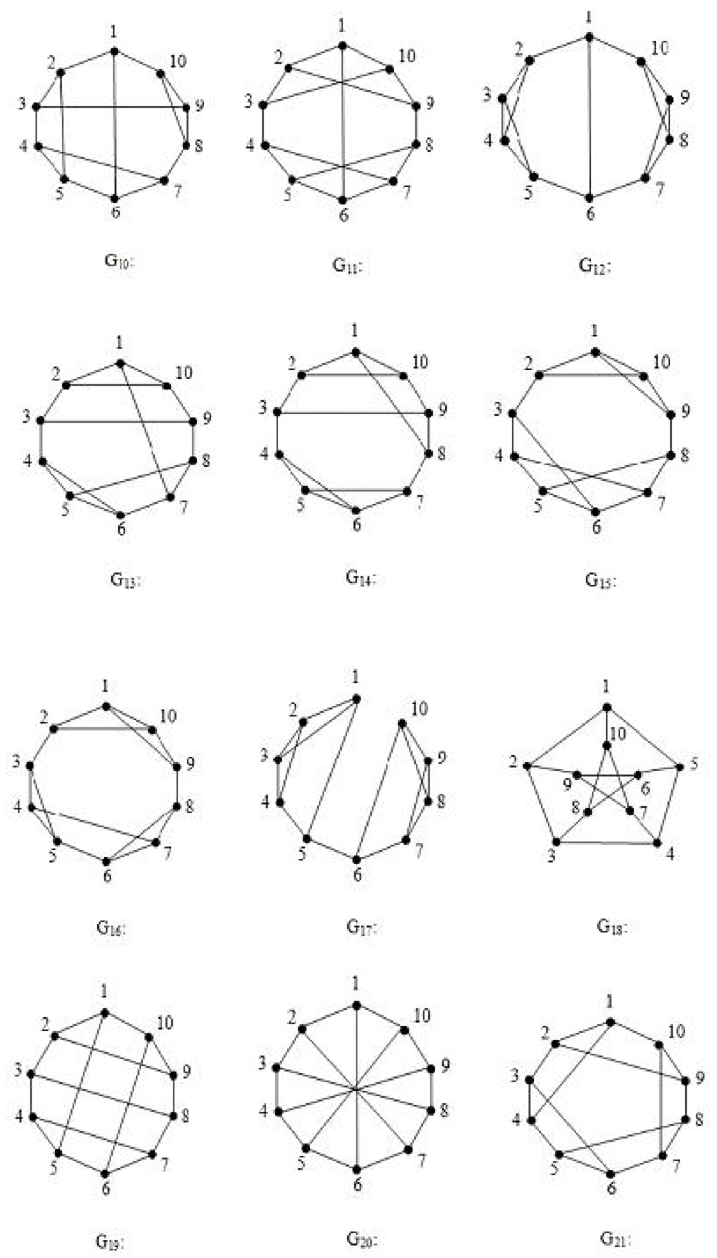

Figure 2: G_{10} to G_{21}.

We prove that G_{18} is the only cubic graph which forms a PBIBD.

Theorem

The graphs $G_i = 1 \leq i \leq 21$, $i \neq 18$ are not PBIB graphs.

Proof. We prove the theorem for the graphs G_1, G_2 *and* G_{19} and proofs are similar for the remaining cases.

In G_1 , there are exactly two γ_t- sets containing 1 and there are three γ_t- sets containing 4 and hence $G_1$is not a PBIB graph.

In G_2there are exactly two γ_t - sets containing 1 and there are three γ_t - sets containing 3.

In G_{19} there are exactly eight γ_t- sets containing 1 and there are seven $\gamma_t$sets containing 3.

Thus each vertex is not appearing in a fixed number of blocks ; hence these will not form PBIBD.

Theorem

The only graph which forms a PBIBD is G_{18}

Proof. There exists a parameters of first kind as (10,10,4,4,2,1) and parameters of second kind as

$P^1 = \begin{pmatrix} 0 & 2 \\ 2 & 4 \end{pmatrix}$ and $P^2 = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$

$n_1 = 3, \lambda_1 = 2$ and $n_2 = 6, \lambda_2 = 1$

Which proves that only connected cubic graph on ten vertices that is G_{18} forms a PBIBD.

Conclusion & Scope

In this paper we consider (v, b, r, k) - design over cycles and enumerating the minimum total dominating sets of cycles and paths. One may also consider the designs whose block are subsets of vertex set of regular graph with given property such as vertex cover, edge independent sets and many other properties associated with edge set and vertex set of graph. Exploration of designs of this sort may provide considerable insight in to the construction of designs and even may lead to the construction of some special types of codes in coding theory.

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