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Observations on the Non-homogeneous Quintic Equation with Four Unknowns

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Abstract

We obtain infinitely many non-zero integer quadruples (x, y, z, w) satisfying the quintic equation with four unknowns. $xy + 6z^2 = (k^2 + 5)^n w^5$. Various interesting relations between the solutions and special numbers, namely, polygonal numbers, pyramidal numbers, Jacobsthal numbers, Jacobsthal-Lucas number, keynea numbers, Four Dimensional Figurative numbers and Five Dimensional Figurative numbers are exhibited.

Keywords: Quintic equation with four unknowns, integral solutions, 2-dimentional, 3-dimentional, 4-dimentional and 5-dimensional Figurative numbers.
MSC 2000 Mathematics subject classification: 11D41.

Notations

$T_{m,n} = n\left(1 + \frac{(n-1)(m-2)}{2}\right)$ -Polygonal number of rank n with size m

$P_n^m = \frac{1}{6}\left(n(n+1)((m-2)n + 5 - m\right)$ -Pyramidal number of rank n with size m

$SO_n = n(2n^2 - 1)$ -Stella octangular number of rank n

$S_n = 6n(n-1) + 1$ -Star number of rank n

$PR_n = n(n+1)$ -Pronic number of rank n

$J_n = \frac{1}{3}\left(2^n - (-1)^n\right)$-Jacobsthal number of rank n

$j_n = 2^n + (-1)^n$ -Jacobsthal-Lucas number of rank n

$KY_n = (2^n+1)^2 - 2$ -keynea number.

$F_{5,n,3} = \frac{n(n+1)(n+2)(n+3)(n+4)}{5!}$ = Five Dimensional Figurative number of rank n
whose generating polygon is a triangle.

$F_{4,n,3} = \frac{n(n+1)(n+2)(n+3)}{4!}$ = Four Dimensional Figurative number of rank n
whose generating polygon is a triangle

Introduction

The theory of diophantine equations offers a rich variety of fascinating problems. In particular,quintic equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity[1-3]. For illustration, one may refer [4-10] for quintic equations with three,four and five unknowns. This paper concerns with the problem of determining non-trivial integral solutions of the non-homogeneous quintic equation with four unknowns given by $xy + 6z^2 = (k^2+5)^n w^5$.A few relations between the solutions and the special numbers are presented.

Method of Analysis

The Diophantine equation representing the quintic equation with four unknowns under consideration is

$$xy + 6z^2 = (k^2+5)^n w^5 \tag{1}$$

Introduction of the transformations

$$x = u+v, y = u-v, z = v \tag{2}$$

in (1) leads to

$$u^2 + 5v^2 = (k^2+5)^n w^5 \tag{3}$$

The above equation (3) is solved through three different approaches and thus, one obtains three distinct sets of solutions to (1)

Approach 1:

Let $w = a^2 + 5b^2$ (4)

Substituting (4) in (3) and using the method of factorisation, define

$$(u + i\sqrt{5}v) = (k + i\sqrt{5})^n (a + i\sqrt{5}b)^5$$

$$= r^n \exp(in\theta)(a + i\sqrt{5}b)^5$$

where $r=\sqrt{k^2+5}, \theta=\tan^{-1}\frac{\sqrt{5}}{k}$ (5)

Equating real and imaginary parts in (5) we get

$$u=r^n[\cos n\theta(a^5-50a^3b^2+125ab^4)-\sqrt{5}\sin n\theta(5a^4b-50a^2b^3+25b^5)]$$

$$v=r^n[\cos n\theta(5a^4b-50a^2b^3+25b^5)]+\frac{\sin n\theta}{\sqrt{5}}(a^5-50a^3b^2+125ab^4)]$$

In view of (2) and (4), the corresponding values of x, y, z and w are represented by

$$\left.\begin{aligned}
&x=r^n[\cos n\theta(a^5-50a^3b^2+125ab^4+5a^4b-50a^2b^3+25b^5)+\\
&\frac{\sin n\theta}{\sqrt{5}}(a^5-50a^3b^2+125ab^4-5(5a^4b-50a^2b^3+25b^5)]\\
&y=r^n[\cos n\theta(a^5-50a^3b^2+125ab^4-5a^4b+50a^2b^3-25b^5)-\\
&\frac{\sin n\theta}{\sqrt{5}}(a^5-50a^3b^2+125ab^4+5(5a^4b-50a^2b^3+25b^5)]\\
&z=r^n[\cos n\theta(5a^4b-50a^2b^3+25b^5)+\frac{\sin n\theta}{\sqrt{5}}(a^5-50a^3b^2+125ab^4)\\
&w=a^2+5b^2
\end{aligned}\right\} \quad (6)$$

Properties

1. $x(a,b)-y(a,b)-2z(a,b)=0$
2. $z(a,a)+r^n(20\cos n\theta-\frac{76\sin n\theta}{\sqrt{5}})(4T_{3,a-1}.P_a^5+6P_a^3-6T_{3,a}+PR_a-T_{4,a})=0$
3. $(w(2^n,1)-3J_{4n})$ is a nasty number.
4. $2(w(2^n,1)-j_{4n})$ is a cubical integer

Remark 1

To analyse the nature of the solutions, one has to go for particular values to n.
For simplicity and clear understanding, taking $n=0$ in (6), the corresponding values of the integer quadruples (x, y, z, w) satisfying $xy+6z^2=w^5$ are represented by

$$\left.\begin{aligned}
&x=a^5-50a^3b^2+125ab^4+5a^4b-50a^2b^3+25b^5\\
&y=a^5-50a^3b^2+125ab^4-5a^4b+50a^2b^3-25b^5\\
&z=5a^4b-50a^2b^3+25b^5\\
&w=a^2+5b^2
\end{aligned}\right\} \quad (7)$$

The integer quadruples (x, y, z, w) represented by (7) satisfies the following properties:

1. $x(a,b)-y(a,b)-10T_{4,a^2}+200T_{3,a}\equiv 0(\text{mod}\,50)$

2. $240F_{5,a,3}-480F_{4,a,3}-300P_a^3+540T_{3,a}+304T_{4,a}-152T_{6,a}-x(a,1)-y(a,1)=0$

3. $30[z(a,a)+y(a,a)-x(a,a)-2400F_{5,a,3}+900P_a^4+700T_{3,a}]$ is a nasty number.

4. $4[y(1,b)-z(1,b)+w(1,b)-250T_{3,b^2}+85T_{6,b}\times 255T_{4,b}-170T_{5,b}]$ is a cubical integer.

5. $8[x(1,b)+y(1,b)-500T_{3,b^2}+350T_{4,b}]$ is a biquadratic integer.

6. $z(2^n,1)-w(2^n,1)-66=5j_{4n}-51j_{2n}$

7. $x(2^n,1)-y(2^n,1)+z(2^n,1)+90=15KY_{2n}-540J_{2n}$

8. If (x_0,y_0,z_0,w_0) is any given solution of (1) then each of the following quadruples
satisfies (1):

$(i)(6x_0-25y_0-60z_0,-x_0+6y_0+12z_0,x_0-5y_0-11z_0,w_0)$

$(ii)(-3x_0+8y_0+24z_0,2x_0-3y_0-12z_0,-x_0+2y_0+7z_0,w_0)$

$(iii)(-6x_0+49y_0+84z_0,x_0-6y_0-12z_0,-x_0+7y_0+13z_0,w_0)$

$(iv)(-2x_0+3y_0+12z_0,3x_0-2y_0-12z_0,-x_0+y_0+5z_0,w_0)$

$(v)(3x_0-2y_0-12z_0,-2x_0+3y_0+12z_0,x_0-y_0-5z_0,w_0)$

Remark 2

It is also worth mentioning here that we get integer solutions when $n=1,2,3,........$ in (1).

For the sake of understanding we exhibit below the integral solutions corresponding to

$n=1,k=1,2$

$n=2,k=1,2$

Exhibit 1: $n=1,k=1$

$x=2a^5-100a^3b^2+250ab^4-20a^4b+200a^2b^3-100b^5$

$y=-30a^4b+300a^2b^3-150b^5$

$z=5a^4b-50a^2b^3+25b^5+a^5-50a^3b^2+125ab^4$

$w=a^2+5b^2$

Exhibit 2: $n=1,k=2$

$x=3a^5-150a^3b^2+375ab^4-15a^4b+150a^2b^3-75b^5$

$y=a^5-50a^3b^2+125ab^4-35a^4b+350a^2b^3-175b^5$

$z=10a^4b-100a^2b^3+50b^5+a^5-50a^3b^2+125ab^4$

$w=a^2+5b^2$

Exhibit 3: $n=2, k=1$

$x=-2a^5+100a^3b^2-250ab^4-70a^4b+700a^2b^3-350b^5$

$y=(-6)[a^5-50a^3b^2-125ab^4+5a^4b-50a^2b^3+25b^5]$

$z=-20a^4b+200a^2b^3-100b^5+2a^5-100a^3b^2+250ab^4$

$w=a^2+5b^2$

Exhibit 4: $n=2, k=2$

$x=3a^5-150a^3b^2+375ab^4-105a^4b+1050a^2b^3-525b^5$

$y=-5(a^5-50a^3b^2+125ab^4)-19(5a^4b-50a^2b^3+25b^5)$

$z=-5a^4b+50a^2b^3-25b^5+4a^5-200a^3b^2+500ab^4$

$w=a^2+5b^2$

Approach 2

Taking $n=0$ in (3), we have,

$$u^2+5v^2=w^5 \tag{8}$$

whose solution is given by

$u_0=a^5-50a^3b^2+125ab^4$

$v_0=5a^4b-50a^2b^3+25b^5$

Again taking $n=1$ in (3), we have,

$$u^2+5v^2=(k^2+5)w^5 \tag{9}$$

whose solution is represented by

$u_1=ku_0-5v_0$

$v_1=u_0+ku_0$

The general form of integral solutions to (8) is given by

$$\begin{pmatrix} u_s \\ v_s \end{pmatrix}=\begin{pmatrix} A_s & i\sqrt{5}B_s \\ -\frac{i}{\sqrt{5}}B_s & A_s \end{pmatrix}\begin{pmatrix} u_0 \\ v_0 \end{pmatrix},\ s=1,2,3,........$$

where

$$A_s=\frac{(k+\sqrt{5}i)^s+(k-\sqrt{5}i)^s}{2}$$

$$B_s=\frac{(k+\sqrt{5}i)^s-(k-\sqrt{5}i)^s}{2}$$

Thus in view of (2), the following quadruple of integers based on (x_0, y_0, z_0, w_0) also

satisfy (1)

Quadruple: (x_s, y_s, z_s, w_s) ,where

$$x_s = (u_0 + v_0)A_s + i\sqrt{5}(v_0 - \frac{u_0}{5})B_s$$

$$y_s = (u_0 - v_0)A_s + i\sqrt{5}(v_0 + \frac{u_0}{5})B_s$$

$$z_s = v_0 A_s - i\sqrt{5}(\frac{u_0}{5})B_s$$

$$w_s = a^2 + 5b^2$$

The above values of x_s, y_s, z_s satisfy the following recurrence relations respectively

$$x_{s+2} - 2kx_{s+1} + (k^2+5)x_s = 0$$

$$y_{s+2} - 2ky_{s+1} + (k^2+5)y_s = 0$$

$$z_{s+2} - 2kz_{s+1} + (k^2+5)z_s = 0$$

Properties 1

1. $x(a,a,k) + y(a,a,k) - (152A_s - 40i\sqrt{5}B_s)[120F_{5,a,3} - 240F_{4,a,3} + 150P_a^3 - 30T_{3,a} + 2T_{4,a} - T_{6,a}] = 0$
2. $S_a + 30PR_b + 12T_{4,a} - 6SO_a - 60T_{3,b} + 30T_{4,b} - 6w(a,b) = 1$
3. $2x(a,a,k) - 2y(a,a,k) + (40A_s + 152\frac{i}{\sqrt{5}}B_s)[8T_{3,a-1}.P_p^5 + SO_a + \mathrm{P}R_a - T_{4,a}] = 0$

$4z(a,1,k) = A_s(120F_{4,a,3} - 45(OH_a) - 90T_{4,a} - 30T_{3,a} + 25) - \frac{iB_s}{\sqrt{5}}(4T_{3,a-1}.P_a^3 - 294P_a^3 - 76T_{4,a} + 223PR_a) = 0$

5. $2z(a,a,k) + (20A_s + \frac{i}{\sqrt{5}}76B_s)(8T_{3,a-1}.P_a^5 + SO_a + 2T_{3,a} - T_{4,a}) = 0$

Approach 3

Substituting (4) in (3) and using the method of factorisation, define

$$(u + i\sqrt{5}v) = (k + i\sqrt{5})^n (a + i\sqrt{5}b)^5$$

Expanding binomially and equating real and imaginary parts, we have

$$u = f(k)(a^5 - 50a^3b^2 + 125ab^4) - 5g(k)(5a^4 - 50a^2b^3 + 25b^5)$$

$$v = g(k)(a^5 - 50a^3b^2 + 125ab^4) + f(k)(5a^4 - 50a^2b^3 + 25b^5)$$

where

$$\left.\begin{aligned} f(x) &= \sum_{r=0}^{\left[\frac{n}{2}\right]} (-1)^r mc_{2r} k^{n-2r} 5^r \\ g(x) &= \sum_{r=1}^{\left[\frac{n+1}{2}\right]} (-1)^{r-1} nc_{2r-1} k^{n-2r+1} 5^{r-1} \end{aligned}\right\} \qquad (10)$$

In view of (2) and (10) the corresponding integer solution (x, y, z, w) to (1) is obtained as

$$x = [f(k) + g(k)](a^5 - 50a^3b^2 + 125ab^4) + [f(k) - 5g(k)](5a^4 - 50a^2b^3 + 25b^5)$$
$$y = [f(k) - g(k)](a^5 - 50a^3b^2 + 125ab^4) - [f(k) + 5g(k)](5a^4 - 50a^2b^3 + 25b^5)$$
$$z = g(k)(a^5 - 50a^3b^2 + 125ab^4) + f(k)(5a^4 - 50a^2b^3 + 25b^5)$$
$$w = a^2 + 5b^2$$

Conclusion

In conclusion, one may get different patterns of solutions to (1) and their corresponding properties.

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