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A Common Fixed Point Theorem for Six Self Mappings under *D**–compatibility

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Abstract

The object of this paper is to introduce the concept of *D** - compatibility of pair of self maps in a metric space. Using this concept we try to establish a unique common fixed point theorem for six self mappings which satisfy a more general contractive inequality of integral type. The result obtained is a significant generalization of result of general contractive condition of integral type reported earlier and the result obtained is novel.

Keywords: compatible maps, metric space, common fixed point.

Introduction

There are numerous generalizations of the Banach contraction principle. After an interesting result of Kannan [5] many existence theorems dealing with the mappings satisfying various types of contractive conditions. In 1972 Bianchini [1] established a new result by using different contractive condition [5]. Branciari [2] obtained a fixed point theorem for a mapping for an integral type inequality. Rhoades [6] proved two fixed point theorem involving more general contraction conditions. Many authors proved common fixed point theorem using the concept of weakly compatible mapping [4]. The first important result in the theory of fixed point of compatible mappings was obtained by Gerald Jungck in 1986 [3] as a generalization of commuting mappings. Sahu and Sharma [8] introduced the notion of *D*-compatibility in a set X which is not necessarily metric space. Sahu and Dewangan [7] introduced the notation of *D**-compatibility as a generalization of *D*-compatibility in d-topological spaces and established some fixed point theorem1s. It is important to note that every *D*-compatible pair is *D**-compatible but converse is not true. [**Example 1**] One can easily observe that the notion of *D**-compatibility is an important generalization of various known commuting and non-commuting mappings. The work reported here is the generalization of P. Vijayaraju, B.E. Rhoades and R. Mohanraj [9]. A fixed point theorems for a map satisfying a general contractive condition of Integral type.

Preliminaries

Definition

Let A and S be two mappings from a metric space (X, d) into itself. The mappings A and S are said to be compatible if d $(ASx_n, SAx_n)=0$ whenever $\{x_n\}$ is a sequence in X such that $Sx_n=Ax_n=x$, for some $x \in X$.

Definition

Let X be a non empty set and A, S: X→X be two self mappings. Then {A, S} is said to be *D*–compatible if Au=Su for some $u \in X \Rightarrow ASu=SAu$.

Definition

The self mappings A and S defined on a metric space (X, d) is said to be *D**–compatible if Au=Su for some $u \in X \Rightarrow d(A^2u, Au)=d(S^2u, Su)$.

The following example shows that *D** -compatibility is more general than *D*-compatibility.

Example1. Let X= [0, 2] with usual metric. Let A, S: X→X be a mappings defined as

Ax=2-x, if $0\le x\le 1$ and Sx=x+1, if $0\le x\le 1$

=0, if $1\le x\le 2$=2, if $1\le x\le 2$.

Then x=½ is a coincidence point of A and S, So $d(S^2½, S½)=d(A^2½, A½)=1$.

Then {A, S} is *D**-compatible. But AS½ ≠ SA½. So {A, S} is not *D*-compatible.

Proposition1

Every *D*-compatible pair is *D**-compatible.

Proof

Let (X, d) be a metric space and A, S: X→X be two self mappings. Let the pair {A, S} be compatible. Then Au=Su for some $u \in X \Rightarrow ASu=SAu$. Hence

$d(A^2u, Au)=d(ASu, Au)$

$=d(SAu, Su)$

$=d(S^2u, Su)$.

Therefore, {A, S} is *D**-compatible.

Example above shows that the converse of proposition is not true in general.

Result

Theorem.

Let A, B, S, T, P and Q be a six self mappings from metric space (X, d) into itself satisfying the following conditions:

$S(X) \subset PQ(X)$, $T(X) \subset AB(X)$;

Pair (S, AB) and (T, PQ) are *D**-compatible;

Either AB or PQ is complete subspace of X;

AB =BA, PQ=QP, SB= BS, TQ= QT;

$\int_0^{d(Sx,Ty)} \varphi(t)dt \leq \psi\left(\int_0^{M(ABx,PQy)} \varphi(t)dt\right)$

for all x, y ∈ X, where

$M(ABx, PQy) = \max\{d(ABx, PQy), d(Sx, ABx), d(Ty, PQy), \frac{d(Sx,PQy)+d(Ty,ABx)}{2}\}$,

Where $\varphi \in \Phi$, $\psi \in \Psi$. Then A, B, S, T, P and Q have a unique common fixed point in X.

Proof

Let x_0 be any point in X. From (3.1.1) there exists $x_1, x_2 \in X$ such that

$Sx_0=PQx_1$, and $Tx_1=ABx_2$.

Inductively, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$y_{2n} = Sx_{2n} = PQx_{2n+1}$

and $y_{2n+1}=Tx_{2n+1}=ABx_{2n+2}$. for n=1, 2, 3, … .

Put $x=x_{2n}$ and $y=x_{2n+1}$ in (3.1.5), we have

$\int_0^{d(Sx_{2n},Tx_{2n+1})} \varphi(t)dt = \int_0^{d(y_{2n},y_{2n+1})} \varphi(t)dt$

$\leq \psi\left(\int_0^{M(ABx_{2n},PQx_{2n+1})} \varphi(t)dt\right)$

$\leq \psi\left(\int_0^{d(y_{2n-1},y_{2n})} \varphi(t)dt\right)$

Put $x=x_{2n+1}$ and $y=x_{2n+2}$ in (3.1.5), we have

$\int_0^{d(Sx_{2n+1},Tx_{2n+2})} \varphi(t)dt = \int_0^{d(y_{2n+1},y_{2n+2})} \varphi(t)dt$

$\leq \psi\ (\int_0^{M\ (ABx_{2n+1},PQx_{2n+2})} \varphi(t)dt\ )$

$$\leq \psi\ (\int_0^{d\ (y_{2n},y_{2n+1})} \varphi(t)dt\ ).$$

Set $d_0 = \int_0^{d\ (y_0,y_1)} \varphi(t)dt$. Therefore for each $n \geq 0$,

$\int_0^{d\ (y_n,y_{n+1})} \varphi(t)dt \leq\ \psi\ (\int_0^{d\ (y_{n-1},y_n)} \varphi(t)dt\ ) \leq \psi^n(d_0).$

Let m, n ∈ N. Using Triangular inequality, we get

$d\ (y_n, y_m) \leq \sum_{i=1}^{m-1} d\ (y_i\ , y_{i+1}).$

It can be show by induction that

$\int_0^{d\ (y_n,y_m)} \varphi(t)dt\ \leq \sum_{i=1}^{m-1} \int_0^{d\ (y_i,y_{i+1})} \varphi(t)dt\ .$

Then $\int_0^{d\ (y_n,y_m)} \varphi(t)\ dt\ \leq\ \sum_{i=1}^{m-1} \psi^i\ \ (d_0) \leq \sum_{i=1}^{\infty} \psi^i\ (d_0).$

Taking limit as n, m→∞ and using condition for (3.1.4), it follow that $\{y_n\}$ is a cauchy sequence in X. Suppose that AB (X) is complete. So $\{y_{2n}\}$ converges to a point z ∈ X, I,e., z=ABu for some u ∈ X. Also its subsequence converges to same point z. i.e.,

$\{Sx_{2n}\} \rightarrow z$ and $\{Tx_{2n+1}\} \rightarrow z$

$\{PQx_{2n+1}\} \rightarrow z$ and $\{ABx_{2n+2}\} \rightarrow z.$

Put x=u and $y=x_{2n+1}$ in (3.1.5), we have

$\int_0^{d\ (Su,Tx_{2n+1})} \varphi(t)dt \leq \psi\ (\int_0^{M\ (ABu,PQx_{2n+1})} \varphi(t)dt\ )$

Where $M\ (ABu, PQx_{2n+1})$=max {d (ABu, PQx_{2n+1}), d (Su, ABu), d (Tx_{2n+1},PQx_{2n+1}),

$$\frac{d(Su,PQx_{2n+1})+d\ (Tx_{2n+1},ABu)}{2}\ \},$$

Taking limit as n, m→∞, we have

$M\ (ABu, z)=\max\{d\ (z, z), d\ (Su, z), d\ (z, z), \frac{d(Su,z)+d\ (z,z)}{2}\ \}$

$=\max\{0, d\ (Su, z), 0, \frac{d(Su,z)+\ 0}{2}\ \}=\max\{d\ (Su, z), \frac{d(Su,z)}{2}\ \},$

$\leq d\ (Su, z).$

Thus, $\int_0^{d\ (Su,z)} \varphi(t)\ dt \leq \psi\ (\int_0^{d\ (Su,z)} \varphi(t)\ dt).$

Which implies that Su=z. So, Su=ABu=z.

Since S (X) ⊂ PQ (X), there exists a point v ∈ X such that Su=PQv=z.

Put x=u and y=v in (3.1.5), we have

$$\int_0^{d\,(Su,Tv)} \varphi(t)dt \le \psi\left(\int_0^{M\,(ABu,PQv)} \varphi(t)dt\right)$$

Where $M\,(ABu, PQv)$=max $\{d\,(ABu,\ PQv),\ d\,(Su,\ ABu),\ d\,(Tv, PQv),\ \frac{d(Su,PQv)+d\,(Tv,ABu)}{2}\}$,

Taking limit as n, m→∞, we have

$M\,(z,\ PQv)=\max\{d\,(z, z), d\,(z, z), d\,(Tv, PQv), \frac{d(z,PQv)+d\,(Tv,z)}{2}\}$

$=\max\{d\,(Tv, z), \frac{0+d\,(Tv,z)}{2}\}=\max\{d\,(Tv, z), \frac{d\,(Tv,z)}{2}\}=\max\{d\,(Tv, z), \frac{d\,(Tv,z)}{2}\}$

$=\max\{d\,(Tv, z), \frac{d\,(Tv,z)}{2}\}$

$\le d\,(Tv, z)$.

Thus, $\int_0^{d\,(Tv,z)} \varphi(t)dt \le \psi\left(\int_0^{d\,(Tv,z)} \varphi(t)\,dt\right)$.

Which implies that Tv=z. So, Tv =PQv=z.

Since {T, PQ} is *D**-compatible and Tv=PQv=z, it follow that

$d\,(Tz, z)=d\,(TTv, Tv)=d\,(PQ^2v, PQv)=d\,(PQz, z)$.

Put x=u and y=z in (3.1.5), we have

$\int_0^{d\,(Su,Tz)} \varphi(t)dt \le \psi\left(\int_0^{M\,(ABu,PQz)} \varphi(t)dt\right)$ for all x, y ∈ X .

Where $M\,(ABu, PQz)$=max $\{d\,(ABu,\ PQz),\ d\,(Su,\ ABu),\ d\,(Tz,\ PQz),\ \frac{d(Su,PQz)+d\,(Tz,ABu)}{2}\}$,

$M\,(ABu, PQz)=\max\{d\,(ABu, PQz), d\,(z, z), d\,(Tz, PQz), \frac{d(z,PQz)+d\,(Tz,ABu)}{2}\}$,

$=\max\{d\,(z, PQz), d\,(Tz, PQz), \frac{d(z,PQz)+d\,(Tz,z)}{2}\}$,

$=\max\{d\,(z, PQz), d\,(Tz, PQz), \frac{d(z,PQz)+d\,(Tz,z)}{2}\}$,

$=d\,(PQz, z)$

So, $\int_0^{d\,(Su,Tz)} \varphi(t)dt = \psi\left(\int_0^{M\,(ABu,PQz)} \varphi(t)dt\right) \le \psi\left(\int_0^{d\,(PQz,z)} \varphi(t)dt\right)$.

Which implies that PQz=z and Tz=z.

Next, Since Su=ABu=z, then From D^* -compatibility of {S, AB}, we have

d (Sz, z)=d (S^2u, Su)=d (PQ^2u, PQu)=d (PQz, z)

Put x=z and y=v in (3.1.5), we have

$\int_0^{d\,(Sz,Tv)} \varphi(t)dt \leq \psi\left(\int_0^{M\,(ABz,PQv)} \varphi(t)dt\right)$ for all x, y ∈ X .

Where $M\,(ABz,PQv)$=max{d (ABz, PQv), d (Sz, ABz), d (Tv, PQv), $\frac{d(Sz,PQv)+d\,(Tv,ABz)}{2}$ },

$M\,(ABz,PQv)$=max{d (ABz, z), d (Sz, z), d (z, z), $\frac{d(Sz,z)+d\,(z,ABz)}{2}$ },

=max{d (ABz, z), d (Sz, z) + d (ABz, z), $\frac{d(Sz,z)+d\,(z,ABz)}{2}$ },

=d (ABz, z)

So, $\int_0^{d\,(Sz,Tv)} \varphi(t)dt = \psi\left(\int_0^{M\,(ABz,PQv)} \varphi(t)dt\right) \leq \psi\left(\int_0^{d\,(ABz,z)} \varphi(t)dt\right)$.

Which implies that ABz=z and Sz=z.

Hence PQz=ABz=Sz=Tz=z.

Put x=z and y=Qz in (3.1.5), we have

$\int_0^{d\,(Sz,TQz)} \varphi(t)dt \leq \psi\left(\int_0^{M\,(ABz,PQQz)} \varphi(t)dt\right)$ for all x, y ∈ X .

Where $M\,(ABz,PQQz)$=max{d (ABz,PQQz), d (Sz,ABz), d (TQz,PQQz), $\frac{d(Sz,PQQz)+d\,(TQz,ABz)}{2}$ },

Since PQ=QP, TQ= QT

PQ (Qz)=QP (Qz)=Q (PQz)=Qz

TQz=QTz=Qz.

$M\,(ABz,PQQz)$=max{d (z, Qz), d (z, z), d (Qz, Qz), $\frac{d(z,Qz)+d\,(Qz,z)}{2}$ },

=max{d (z, Qz), 0, 0, $\frac{d(Sz,z)+d\,(Qz,z)}{2}$ },

≤d (Qz, z).

So, $\int_0^{d\,(Qz,z)} \varphi(t)dt = \psi\left(\int_0^{M\,(ABz,PQQz)} \varphi(t)dt\right) \leq \psi\left(\int_0^{d\,(Qz,z)} \varphi(t)dt\right)$.

Which implies that Qz=z

Also we have PQz=z, hence Pz=z.

Therefore PQz=Pz=Qz=z.

Now Put x=Bz and y=z in (3.1.5), we have

$\int_0^{d\,(SBz,Tz)} \varphi(t)dt \leq \psi\,(\int_0^{M\,(ABBz,PQz)} \varphi(t)dt\,)$ for all x, y ∈ X .

Where $M\,(ABBz, PQz)$=max{d (ABBz,PQz), d (SBz,ABBz), d (Tz,PQz), $\frac{d(SBz,PQz)+d\,(Tz,ABBz)}{2}$ },

Since AB=BA, SB=BS

AB (Bz)=BA (Bz)=B (ABz)=Bz

SBz=BSz=Bz.

$M\,(ABBz, PQz)$=max{d (Bz, z), d (Bz, Bz), d (z, z), $\frac{d(Bz,z)+d\,(z,Bz)}{2}$ },

=max{d (Bz, z), 0, 0, $\frac{d(Bz,z)+d\,(Bz,z)}{2}$ },

≤d (Bz, z).

So, $\int_0^{d\,(Bz,z)} \varphi(t)dt = \psi\,(\int_0^{M\,(ABBz,PQz)} \varphi(t)dt\,) \leq \psi\,(\int_0^{d\,(Bz,z)} \varphi(t)dt\,)$.

Which implies that Bz=z

Also we have ABz=z, hence Az=z.

Therefore ABz=Az=Bz=z.

Thus we have Az=Bz =Sz=Pz=Qz =Tz=z.

i.e., z, is common fixed point of A, B, P, Q, S and T.

Uniqueness

Let w be another common fixed point of A, B, P, Q, S and T.

Then Aw=Bw=Pw=Sw=Tw=Qw=w.

Put x=z and y=w in (3.1.5) we have

$\int_0^{d\,(Sz,Tw)} \varphi(t)dt \leq \psi\,(\int_0^{M\,(ABz,PQw)} \varphi(t)dt\,)$ for all x, y ϵ X .

Where $M\,(ABz, PQw)$=max {d (ABz, PQw), d (Sz, ABz), d (Tw, PQw), $\frac{d(Sz,PQw)+d\,(Tw,ABz)}{2}$ },

$M\,(ABz, PQw)$=max {d (z, w), d (z, z), d (w, w), $\frac{d(z,w)+d\,(w,z)}{2}$ }=d (z, w)

So, $\int_0^{d\,(Sz,Tw)} \varphi(t)dt = \psi\,(\int_0^{M\,(ABz,PQw)} \varphi(t)dt\,) \leq \psi\,(\int_0^{d\,(z,w)} \varphi(t)dt\,)$.

Which implies that z=w. Therefore, z is the unique common fixed point of A, B, P, Q, S and T.

In the same manner it can be show that z is the unique common fixed point of A, B, P, Q, S and T when PQ is complete.

References

[1] Bianchini, R.M.T., 1972, "Su un Problema di S. Reich riguardante la teoria dei punti fissi," Boll. Un. Mat. Ital., 5, pp. 103–108.

[2] Branciari, A., 2002, "A fixed point theorem for mappings satisfying a general contractive condition of integral type," Int. J. Math. Sci., 29, pp. 531–536.

[3] Jungck, G., 1986, "Compatible mappings and common fixed points," Internat. Math. Math. Sci., 9, pp. 771–779.

[4] Jungck, G., and Rhoades, B.E., 1998, "Fixed point for set valued functions without continuity," Indian J. Pure Appl. Math., 29 (3), pp. 227-238.

[5] Kannan, R., 1969, "Some results on fixed point-II," Amer. Math. Monthly., 76, pp. 405–408.

[6] Rhoades, B.E., 2003, "Two fixed point theorems for mappings satisfying a general contractive condition of integral type," Int. J. Math. Math. Sci., 63, pp. 4007–4013.

[7] Sahu, D.R., and Dewangan, C.L., 1997, "Fixed point theorems in d -complete topological spaces," National Seminar on Functional Analysis, pp. 24–32.

[8] Sharma, B.K., and Sahu, D.R., 1997, "Hybrid D-compatibility for solution functional equations," Indian Acad. Math., 18, pp. 101–114.

[9] Vijayaraju, P., Rhoades, B.E., and Mohanraj, R., 2005, "A fixed point theorem for maps satisfying a general contractive condition of integral type," Int. J. Math. Math. Sci., 15, pp. 2359–2364.