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U-Compactness and G_δ-Continuity

P. Jha [1] , B.P. Tripathi [2] and Manisha Shrivastava [3]

[1]*Govt . J. Y . Chhattisgarh College , Raipur (C.G.) India*
[2]*Govt. Nagarjun Science College , Raipur(C.G.) India*
[3]*B. R .S. M . College of Agricultural Engineering and Technology, Raipur (C.G.) India*
[3]*Email: shrivastavamanisha9@gmail.com*

Abstract

Velleman proved that a mapping from R to R is continuous if and only if the images of compact sets are compact sets . Arenas and Puertas generalized this characterization of cont--inuity between two topological spaces . G. Nordo and Pasinkov generalized the Arenas and Puertas result to some different classes of spaces, by generalizing the definition of set of mappings characterized by images of sets . In the present paper , we prove results based on G_δ-continuity and *U*-compact spaces.We prove that the *U*-compactness of the space is preserved under Gδ - continuous mapping and also prove that G_δ-continuous image of product of two *U*-compact spaces , is again *U*-compact.

Keywords : *U*-compact spaces , G_δ - continuity.
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Introduction

The problem to characterize the continuity by images of sets was studied first [3] by Vellman that have shown that the set C(R,R) of all real continuous function on R can not characterized by images of sets . Arenas and Puertas have generalized Velleman's result by concerning C(R,R) and have proved the following:

Theorem (AP) : Let X be a locally connected first countable space and let y be a regular normal space . Then $C(X,Y) = C_{A,A} \cap C_{B,B}$ where A is the family of all connected sets and B is the family of compact sets. Nordo and Pasinkov generalized Theorem AP to some different classes of spaces like continuity on q-spaces , continuity on sequential spaces and they prove the theorem :

Let $f : X \rightarrow Y$ be a Gδ - continuous mapping from a locally connected regular q-space

X to normal space Y . If f maps countable compact sets in countable compact sets and connected sets in connected sets then it is continuous.

G_δ - Continuity :

Definition 1.1 : A mapping f: X $\rightarrow$ Y between spaces X and Y is said to be G_δ-continuous at a

point $x \in X$, if for every neighborhood V of fx, there exists a G_δ-set G in X such that $x \in G$ and fG $\subset$ V.

A mapping is called G_δ-continuous if it is G_δ-continuous at any point of the domain. The set of all G_δ-continuous mappings from a space X to a space Y will be denoted by C_δ (X,Y) . Obviously C(X,Y) $\subset$ C_δ (X,Y) (where C(X,Y) consists of all continuous mappings from X to Y) .

There has been much work on different types of compactness and continuity in general topology. But up till now no one has worked on *U*-compactness with G_δ-continuity. In the present paper we prove various results on *U*-compactness and G_δ-continuity.

***U*-compactness : [1]**

Definition 1.2 : Let *U*={*A*} be any collection of open coverings of a topological space X , containing all finite open coverings as subsystem . The topological space X is *U*-compact, if each open cover of X has a refinement $A \subset U$.

In the section 2 , we prove that , 'a mapping f: X $\rightarrow$ Y is G_δ-continuous if and only if the image of a *U*-compact space X is , *U*-compact .

In the section 3 , we prove that the product of two *U*-compact spaces is again *U*-compact . We also prove that the projection mappings $\pi 1$:X $\times$Y $\rightarrow$ X or $\pi 2$: X$\times$Y $\rightarrow$ Y is G_δ-continuous and also prove that the G_δ-continuous image of product of two *U*-compact spaces is again *U*-compact .

New Characterization of *U*-compactness

Theorem 2.1 : Let f: X $\rightarrow$ Y be a mapping from X to Y then prove that the mapping f is G_δ- continuous if and only if the image of a *U*-compact space X is again *U*-compact.

Proof : Let f: X $\rightarrow$ Y be a mapping between two spaces X and Y and suppose the space X is *U*-compact.

Claim : space Y is *U*-compact .

Let $U^* = \{ V\alpha \}_{\alpha \in \Lambda}$ be the collection of open coverings of Y , containing all the finite open coverings of Y as subsystem , where each $V_\alpha \in U^*$ is the open neighborhood of

some point $f(x_\alpha)$, for all $\alpha \in \Lambda$, $x_\alpha \in X$. Those open coverings of space Y, which are not in U^*, like V_β ,
for all $\beta \in \Lambda_1$ are also the neighborhood of some point $f(x_\beta)$, for all $\beta \in \Lambda_1$, $x_\beta \in X$
Let $U = \{ f^{-1}(V_\alpha)\}$ be the collection of coverings of space X where each $f^{-1}(V_\alpha)$, $\alpha \in \Lambda$ be the open set . Each open covering $f^{-1}(V_\alpha)$, for all α, is also the neighborhood of some point x_α in X . Since the mapping f is G_δ-continuous , so corresponding to neighborhood V_α of $f(x_\alpha)$, for all $\alpha \in \Lambda$, in Y we can have G_δ-set G_α in X and corresponding to the neighborhood V_β of $f(x_\beta)$, for all $\beta \in \Lambda_1$, we can have $G\delta$-set G_β in X . Since X is U-compact space , so each covering of X , should have a refinement in U .Therefore each open covering $f^{-1}(V_\alpha)$, for all α, can have a refinement $G\alpha$, for all α in U , as well as those open coverings $f^{-1}(V_\beta)$, for all β , not belongs to U , also have refinement G_β , for all β in .
Again by using the definition of G_δ-continuity ,we can have $x_\alpha \in G_\alpha$, for all α, such that $f(G\alpha) \subset V_\alpha$, for all $\alpha \in \Lambda$, in U^* . Here $f(G_\alpha)$ and $f(G_\beta)$ have been taken as refinements of open coverings of Y as subsystem . Let $\{V_\beta\}$, for all $\beta \in \Lambda_1$, not belongs to U^* are also the open coverings of Y . Now open coverings V_α , for all $\alpha \in \Lambda$ be the neighborhood of some point $f(x_\alpha)$, and V_β are also the neighborhood of some point $f(x_\beta)$, for all $\beta \in \Lambda_1$. Now let $U = \{f^{-1}(V_\alpha)\}_{\alpha \in p}$ (arbitrary set p) be the collection of the open coverings of X , containing all the finite coverings as subsystem . Let $f^{-1}(V_\beta)$, for all $\beta \in p_1$, are also the open coverings of X not belongs to U .
Here we also assume that $f^{-1}(V_\alpha)$, for all α,are the neighborhood of the points $x_\alpha \in X$ and $f^{-1}(V_\beta)$, for all $\beta \in p_1$, are the neighborhood of the points x_β, , in X . So every x_α , for all α and x_β , for all β are the interior points of $f^{-1}(V_\alpha)$ and $f^{-1}(V_\beta)$ respectively . So we can assume open sets $G\alpha$, for every α, G_β , for every β (as $G\delta$-sets in X) such that $x_\alpha \in G\alpha \subset f^{-1}(V_\alpha)$, for all α and $x_\beta \in G_\beta \subset f^{-1}(V_\beta)$, for all β (Since the collection of open sets all $G\alpha$'s and all G_β's have been taken as refinements of open coverings $f^{-1}(V_\alpha)$ and $f^{-1}(V_\beta)$ respectively of X, in U) in U . This implies that $fx_\alpha \in f(G\alpha) \subset V_\alpha$, for every α and $fx_\beta \in f(G_\beta) \subset V_\beta$, for every β (Here $f(G\alpha)$ and $f(G_\beta)$ are the refinements of the open coverings V_α and V_β respectively of X) in U^* , as space Y is U-compact .

It shows that mapping $f: X \rightarrow Y$ is G_δ-continuous .

Productivity of *U*-Compactness

Theorem 3.1 : The product of two U–compact spaces is again U-compact.

Proof : Step 1: Suppose we have given two spaces X and Y (both X and Y are compact) , with Y is U_2-compact .

Let x_0 be any point of X , and N is any open set in X×Y , containing the slice $x_0 \times Y$ of X×Y . For the product of two compact spaces , if Y is compact , then we know that by the Tube

Lemma , N contains some tube $w \times Y$ about $x_0 \times Y$, w is the neighborhood of x_0 in X. Since Y is the U_2-compact , let $U_2 = \{A_\alpha\}_{\alpha\in\Lambda}$ be the collection of open sets of Y , containing all the finite coverings of Y , as subsystem . Since Y is U_2-compact , so each open covering of Y will have a refinement B_α in U_2 . Let $U^* \times U_2 = \{W \times A_\alpha\}$ be the collection of open coverings of $x_0 \times Y$. Since $x_0 \times Y$ is compact , so is $U^* \times U_2$ compact , therefore each open covering of $x_0 \times Y$ will have a refinement $W_{x0} \times B_\alpha$ in $U^* \times U_2$ (where $W_{x0} \subset W$ is again a neighborhood of x_0.)

Step 2 :- Let us assume that the space X and Y are u_1-compact and U_2-compact respectively.

Let $U_1 \times U_2 = \{a_\alpha \times b_\alpha\}$ be the set of open covering all the finite coverings of X × Y, containing all the finite coverings of X × Y as subsystem . Since $x_0 \times Y$ is $U^* \times U_2$-compact so

it can be covered by many elements of $U_1 \times U_2$ and the union $a_{\alpha 1} \times b_{\alpha 1} \cup a_{\alpha 2} \times b_{\alpha 2} \cup a_{\alpha 3} \times b_{\alpha 3} \cup$

$\cup a_{\alpha m} \times b_{\alpha m} = N$, which is an open set containing $x_0 \times Y$, so by the Tube Lemma the open set N contains a tube $W \times Y$ about $x_0 \times Y$ and $W \times Y$ also covered by many elements of $U_1 \times U_2$ (W is an open set in X) . Since $W \times Y$ is $U_1 \times U_2$-compact , so each open covering of $W \times Y$ will have a refinement in $U_1 \times U_2$.

Thus for each x in X , we can choose a neighborhood W_x of x , such that the tube $W_x \times Y$ can be covered by finitely many elements of $U_1 \times U_2$. The collection of all such neighborhoods W_x is an open covering of X . Since X is U_1-compact , so each open covering of X will have a refinement in $W_x' \subset W_x$ for every x, in U_1 and the union of the tubes $W_1 \times Y$, $W_2 \times Y$, $W_3 \times Y$……. $W_m \times Y$, is the whole X ×Y . Therefore each open covering of X×Y will have a refinement in $U_1 \times U_2 = U$. Therefore X ×Y is U-compact.

Lemma: Prove that the projection mapping $\pi_1 : X \times Y \to X$ or $\pi_2: X \times Y \to Y$ is $G\delta$-continuous .

Proof : Let V_α be a neighborhood of any point $\pi_1 (x_0,y_0)$. Let $G_1 \times Y$ be a $G\delta$-set (Since $G_1 \times Y$ is an open set in X × Y) in X × Y , containing (x_0,y_0) i.e. $(x_0,y_0) \in G_1 \times Y$.

So $\pi_1 (x_0,y_0) \in \pi_1 (G_1\times Y)$. Since V_α be the neighborhood of the point $\pi_1 (x_0,y_0)$.

Therefore $\pi_1 (x_0,y_0) \in \pi_1 (G_1\times Y) \subset V_\alpha$, for all (x_0,y_0) . This shows that the mapping $\pi_1 : X \times Y \rightarrow X$ is Gδ-continuous .

Theorem 3.2 : If the product of two *U*-compact spaces X and Y is *U*-compact and the mapping $\pi_1 : X \times Y \rightarrow X$ is Gδ-continuous , then the π_1 image of product of two *U*-compact spaces is again *U*-compact .

Proof: Let V_α be the neighborhood of any point $\pi_1 (x_0,y_0)$ in X . Let $U_1 = \{ V_\alpha \}_{\alpha\in\wedge}$ be the collection of open coverings of X , containing all the finite open coverings as subsystem.

Let $U_1 \times U_2 = \{ V_\alpha\times Y\}_{\alpha\in\wedge}$ be the set of open coverings of X × Y , containing all the finite open coverings as subsystem . Let $G_1\times G_2$ be any open set (Gδ-set , being an open set) containing the point (x_0,y_0) in $V_\alpha \times Y$, where G_1 is Gδ-set in V_α i.e. in X and G_2 is the Gδ-set in Y .

Since X × Y is *U*-compact , so every open cover $V_\alpha\times Y$ of X × Y , containing refinement suppose

$G_1\times G_2$ in $U_1\times U_2$. Since $(x_0,y_0) \in G_1\times G_2$. This implies that $\pi_1 (x_0,y_0) \subset \pi_1 (G_1\times G_2)$.

Since the mapping is Gδ-continuous , so $\pi_1 (G_1\times G_2) \subset V_\alpha$, for all α. If $\pi_1 (G_1\times G_2)$ can be taken as refinements of open covering V_α , for all α, of X . Thus we can say that every open cover V_α of X having a refinement in U_1 .

Thus the space X is U_1 = *U*-compact .

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