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Twin Primes and the Zeros of the Riemann Zeta Function

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Abstract

The Legendre-type relation for the counting function of ordinary twin primes is reworked in terms of the inverse of the Riemann zeta function. Its analysis sheds light on the distribution of the zeros of the Riemann zeta function in the critical strip and their link to the twin prime problem.

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1. Introduction

The pair sieve for ordinary twin primes [1] leads to a formula for the twin prime counting function $\pi_2(x)$ that is analogous to Legendre's formula [2] for the prime number counting function $\pi(x)$. Before and after separating it into main and error terms [1], it is rewritten here using relevant Dirichlet series. Since the Riemann zeta function ends up in the denominator of the contour integrals, this feature links the zeta zeros to twin primes, much like $\pi(x)$ or related counting functions are expressed as Perron integrals over ζ'/ζ in analytic number theory [3], [4]. Our analysis sheds light on the role of twin primes in the distribution of the nontrivial zeros of the Riemann zeta function, which are those in the critical strip, as usual.

In Sect. 2 the main concepts, such as twin ranks, non-ranks and remnants of the twin-prime pair sieve are recalled along with its main result, the Legendre type formula for π_2. In Sect. 3 it is rewritten as a Perron integral and analyzed. In Sect. 4 the findings are summarized and discussed.

2. Review of the Pair Sieve and Notations

The prime numbers 2, 3 do not play an active role here because they are not of the standard form $6m \pm 1$. This also applies to the first twin prime pair 3, 5. From now on p denotes a prime number or variable and p_j the jth prime with $p_1 = 2, p_2 = 3, p_3 = 5, \ldots$. In our twin prime sieve p_j plays the role of the variable $\sqrt{x}$ in Eratosthenes' sieve.

Definition 2.1. If $6m \pm 1$ is an ordinary twin prime pair for some positive integer m, then m is its *twin rank*. A positive integer n is a *non-rank* if $6n \pm 1$ are **not both** prime.

The arithmetical function $N(x),\ x \neq n + \frac{1}{2}$ is needed for non-ranks.

Definition 2.2. If x is real then $N(x)$ is the integer nearest to x. The ambiguity for $x = n + \frac{1}{2}$ with integral n will not arise.

In Ref. [1] we then prove

Lemma 2.3. If $p \geq 5$ is prime then the positive integers

$$k(n, p)^{\pm} = np \pm N(\frac{p}{6}) > 0,\ n = 0, 1, 2, \ldots \tag{1}$$

are non-ranks. If an integer $k > 0$ is a non-rank, there is a prime $p \geq 5$ so that Eq. (1) holds with either $+$ or $-$ sign.

This means that the pairs $6k^{+} \pm 1$ and $6k^{-} \pm 1$ each contain at least one composite number. Therefore, the primes $p \geq 5$ organize all non-rank numbers in pairs of arithmetic progressions. These pairs are twin prime analogs of multiples $np,\ n > 1$, of primes struck from the integers in Eratosthenes' sieve.

Given a prime $p \geq 5$, when all non-ranks to primes $5 \leq p' < p$ are subtracted from the non-ranks to p, then the non-ranks to **parent** prime p are left forming the set $\mathcal{A}_p$. This process [1] naturally introduces the primorial $L(p) = \prod_{5 \leq p' \leq p} p'$ as the period (of its arithmetic progressions). $L(p_j) \to \infty$ is the twin prime sieve's analog of the variable $x \to \infty$ in Eratosthenes' sieve.

Definition 2.4. Let $p \geq p' \geq 5$ be prime. The supergroup $\mathcal{S}_p = \bigcup_{p' \leq p} \mathcal{A}_{p'}$ contains the sets of arithmetic non-rank progressions of all $\mathcal{A}_{p'},\ 5 \leq p' \leq p$.

The number $S(p)$ counts the non-ranks of $\mathcal{S}_p$ over one period $L(p)$.

Definition 2.5. Since [1] $L(p) > S(p)$, there is a set $\mathcal{R}_p$ of *remnants* r in its first period such that $r \notin \mathcal{S}_p$; they are twin-ranks or non-ranks to primes $p_j < p$, where $p_j \geq 5$ is the jth prime. Let $M(j+1) = \frac{1}{6}(p_{j+1}^2 - 1)$. When all non-ranks to primes $p \leq p_j$ are removed from the first period $[1, L(p_j)]$, all $r \leq M(j+1)$ are twin ranks. These *front* *twin ranks* play the role of the primes $p \leq \sqrt{x}$ in Eratosthenes' sieve that are left over when multiples of primes are removed. The prime p_j is the twin sieve analog of $\sqrt{x}$ there; p_j and $L(p_j)$ correspond to the variable z and $P_z = \prod_{p \leq z} p$, respectively, in more sophisticated sieves.

3. Reworking the Twin Prime Formula

If p_j is the jth prime, then we need

$$L(p_j) = \prod_{5 \leq p \leq p_j} p, \; x = L(p_j) - M(j+1) \tag{2}$$

for the main result of Ref. [1], which is a Legendre-type formula for the number of twin ranks in the first period of length $L(p_j)$ of the supergroup $\mathcal{S}_{p_j}$, where π_2 counts twin pairs below $6x + 1$:

$$\pi_2(6x+1) = R_0 + \sum_{n \leq x, n | L_j(x)} \mu(n) 2^{\nu(n)} \left[\frac{x}{n}\right] + O(1) \tag{3}$$

where $[x/n]$ is the greatest integer function, $L_j(x) = \prod_{p_j < p \leq x} p$, and

$$R_0 = L(p_j) \prod_{5 \leq p \leq p_j} \left(1 - \frac{2}{p}\right) \sim \frac{Cx}{(\log \log x)^2}, \; C > 0[1], \; p_j \sim \log x \to \infty \tag{4}$$

counts the number of remnants in $\mathcal{S}_{p_j}$, that is, twin ranks (prime pairs at distance 2) **and** non-ranks to primes $p_j < p \leq x$. Therefore, the n in the $\sum_n$ of Eq. (3) run over these primes only and their products, and the upper limit is x because the greatest integer function $\left[\frac{x}{n}\right] = 0$ for $n > x$. The twin pair counting function $\pi_2(M(j+1))$ is the number of front twin-ranks and the analog of $\pi(\sqrt{x})$ in Legendre's formula for the prime counting function $\pi(x)$ (see, e.g., pp. 2-3, Ch. 1 of Ref. [5]); they are included in R_0. The error term $O(1)$ in Eq. (3) accounts for the less than perfect cancellation at low values of x of R_0 and the sum in Eq. (3), but Eq. (3) is only relevant at large x in the following. Let us briefly sketch the cancellation of the too large R_0 against the sum in Eq. (3) at large $\log \log x$, upon decomposing $[x/n] = x/n - \{x/n\}$ as usual. Expanding the R_0 product into a sum and combining it with the corresponding sum of the ratios x/n shifts the upper limit of the R_0 sum from p_j to x, when rewritten in its product form. This transforms its entire asymptotics from $\log \log x$ to $\log x$. For more details we refer to Theors. 5.7, 5.8 of Ref. [1]

The asymptotic relations (4) derive from

$$\log L(p_j) = \sum_{5\le p\le p_j} \log p = p_j + R(p_j) = \log x + O\left(\frac{\log^2 x}{x}\right), \tag{5}$$

where the error term comes from $M(j+1)$, and $R(p_j)$ is the remainder of the prime number theorem.

The Dirichlet series characteristic of twin primes and associated with R_0 are

$$P_j(s) = \prod_{p>p_j}\left(1-\frac{2}{p^s}\right) = \prod_{p\le p_j}\left(1-\frac{2}{p^s}\right)^{-1}\sum_{n=1}^{\infty}\mu(n)2^{\nu(n)}n^{-s}. \tag{6}$$

They converge absolutely for $\sigma > 1$, as is evident from the majorant [6]

$$\frac{\zeta^2(s)}{\zeta(2s)} = \sum_{n=1}^{\infty}\frac{2^{\nu(n)}}{n^s},\ \sigma > 1. \tag{7}$$

Note that $2^{\nu(n)} \sim \log n/\zeta(2)$ in the interval $[1, x]$ on average, which is shown in 4.4.18 of Ref. [3]. The corresponding Dirichlet series for primes is

$$P_0(s) = \prod_{p\ge 2}\left(1-\frac{1}{p^s}\right) = \frac{1}{\zeta(s)}, \tag{8}$$

where $\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$ is the Riemann zeta function, and the analog of R_0 here is $x\prod_{p\le\sqrt{x}}\left(1-\frac{1}{p}\right)$ there.

We now use the Perron formula in essentially the form proved in 4.4.15 of Ref. [3].

Lemma 3.1. Let the Dirichlet series $A(s) = \sum_{n=1}^{\infty}\frac{a_n}{n^s}$ be absolutely convergent for $\sigma = \Re(s) > 1$. Then

$$\sum_{n\le x} a_n = \frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT} A(s)\frac{x^s}{s}ds + O\left(\sum_{n=1, n\ne x}^{\infty}\left(\frac{x}{n}\right)^{\sigma}|a_n|\min\left(1, \frac{1}{T|\log\frac{x}{n}|}\right)\right), \tag{9}$$

where the lhs $\sum_{n\le x}$ means that for $n = x,\ a_n$ is reduced by $1/2$.

Corollary 3.2. For $\sigma > 1$

$$\sum_{n\le x} a_n\left[\frac{x}{n}\right] = \frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT} A(s)\zeta(s)\frac{x^s}{s}ds$$
$$+O\left(\sum_{n=1,n\neq x}^{\infty}\left(\frac{x}{n}\right)^{\sigma}\left(\sum_{d|n}|a_d|\right)\min\left(1,\frac{1}{T|\log\frac{x}{n}|}\right)\right). \tag{10}$$

Proof. This follows from Lemma 3.1 and the proof of 4.4.15 in Ref. [3] using

$$\sum_{N\le x}\sum_{n|N} a_n = \sum_{n\le x} a_n\left[\frac{x}{n}\right],\ A(s)\zeta(s) = \sum_{N=1}^{\infty}\frac{1}{N^s}\sum_{n|N} a_n.\ \diamond \tag{11}$$

■

Lemma 3.3.

$$P_1(s)\zeta^2(s) = (1-\frac{1}{2^s})^{-2}\prod_{p>2}\left(1+\frac{1}{p^s(p^s-2)}\right)^{-1} = \frac{(1-2^{-s})^{-2}}{D(s)} \tag{12}$$

$$D(s) = \prod_{p>2}\left(1+\sum_{\nu=0}^{\infty}\frac{2^{\nu}}{p^{(\nu+2)s}}\right) = 1+\sum_{N=4}^{\infty}\frac{2^{2r_e(N)+2r_o(N)-2\bar{r}_e(N)-2\bar{r}_o(N)}}{N^s} \tag{13}$$

converges absolutely for $\sigma > 1/2$. Here

$$r_e(N) = \sum_{i=1}^{m}\nu_i,\ r_o(N) = \sum_{i=1}^{n}(\mu_i+3),\ \bar{r}_e(N) = \sum_{\nu_i>0}1,\ \bar{r}_o(N) = \sum_{\mu_i>0}1 \tag{14}$$

are additive functions for

$$N = p_{e_1}^{2(\nu_1+1)}\cdots p_{e_m}^{2(\nu_m+1)}p_{o_1}^{2\mu_1+3}\cdots p_{o_n}^{2\mu_n+3} \tag{15}$$

in Eq. (13).

Proof. Substituting in

$$\prod_{p>2}\frac{(1-\frac{2}{p^s})}{(1-\frac{1}{p^s})^2} = \frac{1}{\prod_{p>2}(1+\frac{1}{p^s(p^s-2)})} \tag{16}$$

the expansions

$$\frac{1}{1-\frac{2}{p^s}} = 1+\frac{2}{p^s}+\frac{2^2}{p^{2s}}+\cdots, \tag{17}$$

$$1+\frac{1}{p^{2s}(1-\frac{2}{p^s})}=1+\sum_{\nu=0}^{\infty}\frac{2^\nu}{p^{(\nu+2)s}}, \tag{18}$$

yields Eq. (13) with N of the form in Eq. (15). ■

Thus for $\sigma > 1$

$$\begin{aligned} P_j(s)^{-1} &= \zeta^2(s)(1-\frac{1}{2^s})^2 \prod_{2<p\le p_j}(1-\frac{2}{p^s})\prod_{p>2}\left(1+\frac{1}{p^s(p^s-2)}\right) \\ &= \left(\frac{P_1(s)}{\prod_{2<p\le p_j}(1-\frac{2}{p^s})}\right)^{-1} \end{aligned} \tag{19}$$

with $P_1(s)$ from Eq. (12).

We now apply Cor. 3.2 to $P_j(s)$. This yields the Legendre-type formula **before** it is split into its main and error terms according to Ref. [1] so that the leading asymptotic term is R_0.

Theorem 3.4. For $\sigma > 1,\ R_0 = L(p_j)\prod_{5\le p\le p_j}(1-\frac{2}{p})$ and $x > 0$ from Eq. (2),

$$\begin{aligned} \pi_2(6x+1) &= R_0 + \frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT}\frac{(1-\frac{1}{2^s})^{-2}x^s ds}{s\zeta(s)\prod_{2<p\le p_j}(1-\frac{2}{p^s})D(s)} \\ &+O\left(\frac{\zeta^3(\sigma)x^\sigma}{T}\right)+O\left(\frac{x\log^3 x}{T}\right)+O(1), \end{aligned} \tag{20}$$

with $D(s)$ from Eq. (13) and $T > 0$ at least of order $x^c,\ 0 < c < 1$.

Proof. We replace in Eq. (3) the sum by the Perron integral of Cor. 3.2 with $A(s) = P_j(s)$ using Lemma 3.3 for $P_1(s)$ in conjunction with Eq. (19). Canceling the factor $\zeta(s)$, this yields the Perron integral in Eq. (20).

The Euler product of $D(s)$ in Eq. (12) guarantees no zeros for $\sigma > 1/2$. Note that

$$\sum_{f|n}|\mu(f)|2^{\nu(f)}=\sum_{f|\tilde n}2^{\nu(f)}=d_3(\tilde n)\le d_3(n), \tag{21}$$

where $\tilde n$ is the product of different prime divisors of n and, for any $f|\tilde n$,

$$d(f)=\sum_{\delta|f}1=2^{\nu(f)} \tag{22}$$

is the divisor function. Thus, we can use the majorant $d_3(n)$ in the error term in Cor. 3.2, where $\zeta^3(s) = \sum_{n=1}^{\infty} d_3(n)n^{-s}$. We split the sum into three pieces as usual (see, e.g., Theor. 4.2.9 of Ref. [3]) with $S_1 = \sum_{n<x/e}$, $S_2 = \sum_{x/e<n<ex}$, $S_3 = \sum_{n>ex}$. For S_1, S_3 we have $|\log(x/n)| \geq 1$. The total contribution due to S_1 and S_3 is at most $\zeta^3(\sigma)x^\sigma/T$, which is the first error term in Eq. (20) with the constant 1 implied by the $O(\cdots)$.

For S_2, we divide the sum into intervals of the type $I_k = [x \pm 2^k x/T, x \pm 2^{k+1}x/T]$ with $2^{k+1}/T < ex$, and a shorter interval at the end if needed. The number of such intervals is $O(\log T)$. The contribution of the sum over such an interval to the remainder of Perron's formula is of order at most

$$\frac{1}{T}\sum_{I_k} d_3(n)\frac{T}{2^k} = \frac{\sum_{I_k} d_3(n)}{2^k}. \tag{23}$$

The length of I_k is of order $2^k x/T$, which is larger than x^{1-c}, since T is at least of order x^c for some $0 < c < 1$.

Now recall the estimate (see Ref. [6], Ch. 12, formula 12.1.4):

$$\sum_{n<y} d_3(n) = yP_2(\log y) + O(y^{2/3}\log y), \tag{24}$$

P_2 being a certain polynomial of degree 2.

It follows that

$$\sum_{I_k} d_3(n) = O\left(\frac{2^k x \log^2 x}{T}\right) + O(x^{2/3}\log x). \tag{25}$$

If we sum over k the contribution of S_2 is at most of order $O\left(x\frac{\log^3 x}{T}\right)$, which gives the second error term in Eq. (20). The interval $(x/e < n < x)$ can be subdivided and treated similarly leading to the same bound. This completes the proof. ∎

Corollary 3.5. The Riemann hypothesis (RH) is incompatible with the twin prime formula (20) of Theor. 3.4.

Proof. Assuming RH, we shift the line of integration in Eq. (20) from $\sigma > 1$ to $\sigma = \frac{1}{2}+\varepsilon$ for any $\varepsilon > 0$ using Cauchy's theorem. Since RH implies the Lindelöf hypothesis [6] (Chapt. 13), we know that

$$\frac{1}{\zeta(s)} = O(|t|^\delta),\ s = \sigma + it,\ \sigma \geq \frac{1}{2} + \varepsilon \tag{26}$$

for some small $\delta > 0$ that may depend on ε. We note that the zeros $s_p = \log 2/\log p$ of $\prod_{p\leq p_j}(1 - 2/p^s)$ cancel the corresponding poles of $D(s)$. Since only $s_3 \approx 0.6309 > 0.5$, we estimate for $\sigma \geq 1 + \varepsilon$

$$\left|\left(1 - \frac{2}{3^s}\right) D(s)\right|^{-1} = |(3^s - 2 + 3^{-s}) \prod_{p \geq 5} [1 + p^{-s}(p^s - 2)^{-1}]|^{-1} = O(1). \quad (27)$$

As $5 \leq p \leq p_j \sim \log x$ in the product $\prod_p (1 - 2/p^s)$, the latter will be at most of order

$$\Big| \prod_{5 \leq p \leq p_j} \left(1 - \frac{2}{p^s}\right) \Big|^{-1} \leq \prod_{5 \leq p \leq p_j} \left(1 - \frac{2}{\sqrt{p}}\right)^{-1} = O(\log x) \quad (28)$$

for $\sigma \geq 1/2 + \varepsilon$ as $p_j \sim \log x \to \infty$. Hence, on $\sigma = \frac{1}{2} + \varepsilon$ the vertical part of the Perron integral in Theor. 3.4 obeys

$$\int_{\frac{1}{2}+\varepsilon-iT}^{\frac{1}{2}+\varepsilon+iT} \frac{(1 - \frac{1}{2^s})^{-2} x^s ds}{s\zeta(s) \prod_{2 < p \leq p_j} \left(1 - \frac{2}{p^s}\right) D(s)} = O(T^\delta x^{\frac{1}{2}+\varepsilon} \log T \log x), \quad (29)$$

with the $\log T$ factor from the integration.

On the horizontal line segments from $\frac{1}{2} + \varepsilon \pm iT$ to $\sigma \pm iT$ the Perron integral is bounded by $O(T^{\delta-1} x^\sigma)$. The factor $1/\log x$ from the integration cancels $\log x$.

The error terms of Theor. 3.4 are slightly smaller than these, respectively, and can be combined with them. Taking $\sigma = 1 + \varepsilon,\ T = x^\alpha$ and equating the exponents of x in both error terms determines $\alpha = \frac{1}{2}$. Therefore, the Perron integral plus error terms in Eq. (20) are of order $O(x^{\varepsilon+(1+\delta)/2})$ and cannot reduce $R_0 \sim Cx/(\log\log x)^2$ to the known bound [5] $O(x/(\log x)^2)$ for $\pi_2(6x + 1)$, q.e.a. ■

We next address the remainder of the twin prime formula (3) after extracting its asymptotic law [1] using the following Perron integral.

Corollary 3.6. Let $A(s)$ be absolutely convergent for $\sigma > 1$, then for $\sigma > 1$

$$\begin{aligned}\sum_{n<x} a_n \left\{\frac{x}{n}\right\} &= \frac{1}{2\pi i} \int_{\sigma-iT}^{\sigma+iT} ds x^s A(s) \Big[\frac{1}{s-1} - \frac{\zeta(s)}{s}\Big] \\ &+ O\left(\sum_{n=1, n \neq x} \left(\frac{x}{n}\right)^\sigma \sum_{d|n} |a_d| \min\left(1, \frac{1}{T|\log \frac{x}{n}|}\right)\right). \end{aligned} \quad (30)$$

Proof. Using

$$\left\{\frac{x}{n}\right\} = \frac{x}{n} - \left[\frac{x}{n}\right] \quad (31)$$

and applying Lemma 3.1 to $xA(s+1)$ for the ratio x/n, integrated along the line $\sigma > 0$, and Cor. 3.2 we obtain the Perron integral in Eq. (30) upon shifting $s \to s-1$ in the first term. Using $|a_n| \le \sum_{d|n} |a_d|$, the error term of Lemma 3.1 combines with that of Cor. 3.2 giving that of Eq. (30). ■

We now apply Cor. 3.6 to $P_j(s)$ which yields the Perron integral for the error term R_E of Ref. [1] after separating formula (3) into its main and error terms so that the main term obeys the proper asymptotic law expected for twin primes [1]. The error term is the same as in Theor. 3.4. For the cancellation involved in getting the proper asymptotics we refer to the discussion below Eq. (4). Clearly, the sum in Eq. (3), represented by the Perron integral in Theor. 3.4, is $-R_0$ plus an asymptotic term $cx/(\log x)^2$, with $c > 0$ calculated in Ref. [1]. Thus it is large, and an application of the contour deformation to the Perron integral in Theor. 3.4 into the known zero-free region of the Riemann zeta function fails to give a small value *unconditionally* because the optimal $a = 0$ cannot be reached at any finite T.

Theorem 3.7. There are constants $a > 0, 0 < b < c, 1 < \alpha < 2$ so that the twin prime remainder takes on the form

$$\begin{aligned}
-R_E &= \sum_{p_j<n<x, n|L_j(x)} \mu(n)2^{\nu(n)}\left\{\frac{x}{n}\right\} + O(1) = O\left(\frac{x^{1+\frac{a}{\log T}}\log^3 T}{T}\right)\\
&+O\left(\frac{x\log^3 x}{T}\right) + O\left(x^{1-\frac{b}{\log T}}(\log T)^3(\log x)^{\alpha}\right) + O\left(x^{1+\frac{a}{\log T}}\frac{(\log T)^2(\log x)^{\alpha}}{T}\right)\\
&= O\left(x\exp(-\sqrt{c\log x})(\log x)^3\right),\ T = \exp(\sqrt{c\log x}).
\end{aligned} \tag{32}$$

Proof. We start from Cor. 3.6 for $P_j(s)$ in conjunction with the error terms of Theor. 3.4:

$$\begin{aligned}
-R_E &= \sum_{p_j<n<x, n|L_j(x)} \mu(n)2^{\nu(n)}\left\{\frac{x}{n}\right\} + O(1)\\
&= \frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT} \frac{(1-\frac{1}{2^s})^{-2}x^s ds}{\zeta(s)\prod_{2<p\le p_j}(1-\frac{2}{p^s})D(s)}\Big[\frac{1}{(s-1)\zeta(s)} - \frac{1}{s}\Big]\\
&+ O\left(\frac{\zeta^3(\sigma)x^{\sigma}}{T}\right) + O\left(x\frac{\log^3 x}{T}\right).
\end{aligned} \tag{33}$$

By Chapt. 3, formula 3.11.8 of Ref. [6] (this also follows from the sharper estimates in Lemma 12.3 of Ref. [7]) there is an absolute constant $c > 0$ so that

$$\frac{1}{|\zeta(s)|} = O(\log(|t|+2)),\ 0 < t_0 \le |t|,\ \delta_t \le \sigma \le 1+\varepsilon,\ \delta_t = 1 - \frac{c}{\log(|t|+2)} \tag{34}$$

and $\frac{1}{|\zeta(s)|} = O(1)$ for $|t| < t_0,\ \sigma \geq \delta_t$. Let R_T be the rectangle joining the vertices

$$1 + \frac{a}{\log T} - iT,\ 1 + \frac{a}{\log T} + iT,\ \delta + iT,\ \delta - iT,\ \delta = 1 - \frac{c}{\log T}. \tag{35}$$

We move the line segment of integration from $\sigma = 1 + a/\log T,\ a > 0$ to the left on the line $\sigma = 1 - b/\log T$ with $a > 0,\ 0 < b < c$ to be chosen later. Then the bounds of $|\zeta(s)|^{-1}$ in Eq. (34) and below hold on the boundary of R_T. The integrand is holomorphic inside and on the rectangle because $\zeta(s)$ does not vanish there and on the vertical line on the left it is of order at most

$$x^{1-\frac{b}{\log T}} \frac{\log T}{|s| \prod_{p \leq p_j} |1 - p^{-1+\frac{b}{\log T}}|}. \tag{36}$$

Since $p \leq p_j \sim \log x$ we know that $p^{-1+b/\log T} \to 0$ as $x \to \infty$ provided T grows with x faster than a power of $\log x$, which will be the case. Then a lower bound for the product will be at least of order $1/(\log p_j)^\alpha$ for any $1 < \alpha < 2$. So the product is at most of order

$$|\prod_{2<p\leq p_j} (1 - \frac{2}{p^s})|^{-1} = O((\log p_j)^\alpha) = O((\log x)^\alpha). \tag{37}$$

Integration over s gives a factor $\log T$. Thus, the integral over the vertical segment is at most of order $O(x^{1-b/\log T}(\log T)^3(\log x)^\alpha)$. Similarly, the integrals over the horizontal segments are at most of order

$$O\left(x^{1+\frac{a}{\log T}} \frac{(\log T)^2(\log x)^\alpha}{T}\right) \tag{38}$$

Putting all this together we obtain the middle section of Eq. (32).

Now we choose $T = \exp(\tau\sqrt{\log x})$ and optimize with respect to a, b, τ under the conditions $a > 0, 0 < b < c$. In the limit we can set $b = c, a = 0, \tau = \sqrt{c}$ and conclude with the bound on the rhs of Eq. (32). A comment on the choice $a = 0$ is in order. The extra convergence factor $s/((s-1)\zeta(s)) - 1 \to 1/\zeta(s) - 1$ at large $|t|$ is the Dirichlet series $\sum_{n\geq 2} \mu(n)n^{-s}$ that converges on $\sigma = 1$ and oscillates rapidly at large $|t|$. This is how the real reduction from $[x/n]$ in Theor. 3.4 to $\{x/n\}$ in Theor. 3.7 plays out analytically. Its presence allows reaching the optimization point $a = 0$ of the error terms representing the Perron integral in Theor. 3.7.

More precisely, Theor. 11.8(B) for any $\sigma > 1$ and Theor. 11.10 for $\frac{1}{2} < \alpha < \sigma < \beta < 1$ in Ref. [6] imply that the number of solutions of $\zeta(s) = 1 + \varepsilon$ for $\sigma > 1$ and $\zeta(s) = 1 - \varepsilon$ for $\frac{1}{2} < \sigma < 1$ and arbitrarily small $\varepsilon > 0$ have positive density in the interval $[0, T]$ on vertical lines that may be chosen as close as one likes to $\sigma = 1$. Note that

$$1 \leq \left|\frac{s}{s-1}\right| = 1 + \frac{\sigma - \frac{1}{2}}{t^2} + O\left(\frac{1}{t^4}\right), \ \sigma > \frac{1}{2},$$
$$\frac{s}{s-1} = 1 + \frac{\sigma - 1}{t^2} - \frac{i}{t} + O\left(\frac{1}{|t|^3}\right), \ s \neq 1.$$

Thus the convergence factor is as small as one likes for large enough $|t|$ in many areas of the region enclosed by the contour and on it. This is not the case with the Perron integral in Theor. 3.4, where the fluctuating x^s has large absolute value $|x^s| = x^\sigma$. This completes the proof. ■

This proves that the (minimal) asymptotic law obtained in Ref. [1] is valid with the remainder smaller than it by any positive power of $\log x$.

4. Summary and Discussion

When the Legendre-type formula for π_2 is reworked into a Perron integral involving $\zeta^{-1}(s)$, the nontrivial zeta zeros are seen to be linked to the twin prime counting function π_2. The asymptotic law of its leading term

$$R_0 = L(p_j) \prod_{5 \leq p \leq p_j} \left(1 - \frac{2}{p}\right) \sim \frac{Cx}{(\log \log x)^2}, \ \log \log x \to \infty \tag{39}$$

with $x = L(p_j) - M(j+1)$, $p_j \sim \log L(p_j) \sim \log x$ requires $\log \log x$ to become large. In contrast, only $\log x$ is large in the prime number theorem [3], [4]. Therefore, the true asymptotic region of twin primes starts much higher up than for primes. Such values of $x > x_0 = e^{e^{e^e}}$ with $\log \log x_0 = e^e = 15.15426\ldots$ are larger than 2208049118 followed by more than 1.6565 million zeros, i.e. are truly titanic numbers. It is clear that present numerical results of nontrivial zeta zeros are far from reaching the asymptotic twin prime realm. This is valid whether or not there are infinitely many twin primes, because R_0 is the number of remnants including twin pairs (i.e. twin ranks) **and** non-ranks to primes $p_j < p < x$. The Perron integral in Theor. 3.4 represents the latter's contributions that will reduce R_0 to $\pi_2(6x + 1)$, R_0 being much larger than known bounds from sieve theory [3], [5] on π_2 that are due to V. Brun, A. Selberg and others. Only nontrivial zeta zeros in the Perron integral can produce terms that reduce R_0 to the proper magnitude. Our first result is that the zeros on the critical line cannot do the job. Despite trillions of initial zeros on the critical line that are relevant for the prime number distribution without asymptotic twin prime attributes, once twin prime asymptotics matter some zeta zeros must move off the critical line toward the borders of the critical strip.

Finally, from the point of view of our twin prime formulas (3), (20) a finite number of twin primes is neither a simple nor natural case, as it would require the cancellation of the leading and all subleading asymptotic terms involving fine-tuning of the large primes ($> p_j$) that organize the non-ranks.

But this never happens because, when the Perron integral in Cor. 3.6 is developed for $\sum_{n<x, n|L_j(x)} \mu(n) 2^{\nu(n)} \{x/n\}$ using $P_j(s)$ in the known zero-free region of the Riemann zeta function, the twin prime theorem near primorial arguments follows, our second result.

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