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The Stability of some Trigonometric Functional Equations

R. Sakthi and V. Vidhya

*Department of Science and Humanities, Saveetha School of Engineering,
Saveetha University, Chennai – 602 105, Tamil Nadu, India.
E-mail: rsakth@gmail.com*

Abstract

The aim of this paper is to investigate the stability problem of the Trigonometric functional equations

$$f(2x) - f(2y) = 2g(x+y)f(x-y)$$

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Introduction

In 1940, S.M.Ulam raised the following stability problem [16]:

Let f be a mapping from group G, to a metric group $G^{'}$ with metric *d(.,.)* such that $d(f(xy),f(x)f(y))\leq \varepsilon$

Then does there exists a group homomorphism L and $\delta_\varepsilon > 0$ such that $d(f(x),L(x))\leq \delta_\varepsilon$

for all $x \in G^{'}$?

This problem was solved affirmatively by Hyers[10] under the assumption that is a Banach space. In 1949 – 1950, this result was generalized by the authors Bourgin[7] and Aoki[1] and since then stability problems of many other functional equations have been investigated[8, 10-12]. In 1990, Szekelyhidi[15] has developed his idea of using invariant subspaces of functions defined on a group or semi group in connection with stability questions for the sine and cosine functional equation.

Baker et.al [4] and Bourgin [6] introduced that if f satisfies the stability inequality $|E_1(f) - E_2(f)| \leq \varepsilon$, then either f is bounded or $E_1(f) = E_2(f)$. This is now frequently referred to as super stability.

The super stability of the cosine functional equation

$$f(x+y) + f(x-y) = 2f(x)f(y)$$

and the sine functional equation

$$f(x)f(y) = f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2$$

are investigated by Baker [5] and Cholewa [8] respectively. The stability of the generalized cosine functional equation has been researched in many papers [2, 3, 9, 11-13] and Kim [14] investigated the stability of the generalized sine functional equation.

The aim of this paper is to study the stability problem of the Trigonometric functional equations

$$f(2x) - f(2y) = 2g(x+y)f(x-y) \qquad (T_1)$$

$$g(2y) - g(2x) = 2f(x+y)f(x-y) \qquad (T_2)$$

In this paper, let $(G, +)$ be an abelian group, $\mathbb{C}$ the field of complex numbers, R the field of real numbers and N the natural numbers. We may assume that f and g are non zero functions and ε is a non-negative real constant, a mapping $\varphi: G \to R$.

Stability of the equation (T_1)

In this section, we investigate the stability of the trigonometric functional equation (T_1).

Theorem

Suppose that $f, g: G \to C$ satisfy the inequality

$$|f(2x) - f(2y) - 2g(x+y)f(x-y)| \leq \varphi(x) \qquad (2.1)$$

for all $x, y \in G$. If f fails to be bounded then f and g satisfy

$$g(2x) + g(2y) = 2g(x+y)f(x-y) \qquad (2.2)$$

Proof

Equation (2.1) can be written as

$$|f(u+v) - f(u-v) - 2g(u)f(v)| \leq \varphi\left(\frac{u+v}{2}\right) \qquad (2.3)$$

Let f be unbounded. Then we can choose a sequence $\{v_n\}$ in G such that $0 \neq |f(v_n)| \to \infty$ as $n \to \infty$. Taking $v = v_n$ in (2.3) we obtain

$$\left|\frac{f(u+v_n)-f(u-v_n)}{2f(v_n)}-g(u)\right| \le \frac{\varphi\left(\frac{u+v_n}{2}\right)}{2|f(v_n)|} \tag{2.4}$$

that is

$$\lim_{n\to\infty}\frac{f(u+v_n)-f(u-v_n)}{2f(v_n)} = g(u) \tag{2.5}$$

for all $u \in G$.

Using (2.3) we have

$$\varphi\left(\frac{u+v+v_n}{2}\right)+\varphi\left(\frac{u+v-v_n}{2}\right) \ge |f(u+v+v_n)-f(u-v-v_n)-2g(u)f(v+v_n)| + |f(u+v-v_n)-f(u-v+v_n)-2g(u)f(v-v_n)| \tag{2.6}$$

so that

$$\left|\frac{f((u+v)+v_n)-f((u+v)-v_n))}{2f(v_n)}+\frac{f((u-v)+v_n)-f((u-v)-v_n))}{2f(v_n)} - 2g(u)\frac{f(v+v_n)-f(v-v_n))}{2f(v_n)}\right| \le \frac{\varphi\left(\frac{u+v+v_n}{2}\right)+\varphi\left(\frac{u+v-v_n}{2}\right)}{2|f(v_n)|}$$

Taking lim as $n \to \infty$ on both sides

$$|g(u+v)+g(u-v)-2g(u)f(v)| \le 0$$

that is

$$|g(2x)+g(2y)-2g(x+y)f(x-y)| \le 0$$

for all $x, y \in G$. Therefore f and g satisfies (2.2).

Corollary

Suppose that $f, g: G \to C$ satisfy the inequality

$$|f(2x)-f(2y)-2g(x+y)f(x-y)| \le \varepsilon$$

for all $x, y \in G$. If f fails to be bounded then f and g satisfy

$$g(2x)+g(2y) = 2g(x+y)f(x-y)$$

Stability of the equation (T_2)

In this section, we investigate the stability of the trigonometric functional equation (T_2).

Theorem

Suppose that $f, g: G \to C$ satisfy the inequality

$$|g(2y)-g(2x)-2f(x+y)f(x-y)| \le \varphi(x) \tag{3.1}$$

for all $x, y \in G$. If f fails to be bounded then f satisfy

$$g(2x)+g(2y)=2g(x+y)g(x-y) \tag{3.2}$$

Proof

Equation (3.1) can be written as

$$|g(u-v)-g(u+v)-2f(u)f(v)| \le \varphi\left(\frac{u+v}{2}\right) \tag{3.3}$$

Let f be unbounded. Then we can choose a sequence $\{v_n\}$ in G such that $0 \neq |f(v_n)| \to \infty$ as $n \to \infty$. Taking $v = v_n$ in (2.3) we obtain

$$\left|\frac{g(u-v_n)-g(u+v_n)}{2f(v_n)} - f(u)\right| \le \frac{\varphi\left(\frac{u+v_n}{2}\right)}{2|f(v_n)|} \tag{3.4}$$

that is

$$\lim_{n\to\infty} \frac{g(u-v_n)-g(u+v_n)}{2f(v_n)} = f(u) \tag{3.5}$$

for all $u \in G$.

Using (3.3) we have

$$\varphi\left(\frac{u+v+v_n}{2}\right)+\varphi\left(\frac{u+v-v_n}{2}\right) \ge |g(u-(v+v_n))-g(u+(v+v_n))-2f(u)f(v+v_n)| + |g(u-(v-v_n))-g(u+(v-v_n))-2f(u)f(v-v_n)| \tag{3.6}$$

so that

$$\left|\frac{g((u-v)-v_n)-g((u-v)+v_n))}{2f(v_n)} + \frac{g((u+v)-v_n)-g((u+v)+v_n))}{2f(v_n)} - 2f(u)\frac{f(v+v_n)-f(v-v_n)}{2f(v_n)}\right| \le \frac{\varphi\left(\frac{u+v+v_n}{2}\right)+\varphi\left(\frac{u+v-v_n}{2}\right)}{2|f(v_n)|}$$

Taking lim as $n \to \infty$ on both sides

$$|f(u+v)+f(u-v)-2f(u)f(v)| \le 0$$

where $f(v) := \frac{f(v+v_n)-f(v-v_n)}{f(v_n)}$

that is

$$|f(2x)+f(2y)-2f(x+y)f(x-y)| \le 0$$

for all $x, y \in G$. Therefore f satisfy (3.2).

Corollary

Suppose that $f, g: G \to C$ satisfy the inequality

$$|g(2y) - g(2x) - 2f(x+y)f(x-y)| \le \varepsilon$$

for all $x, y \in G$. If f fails to be bounded then f satisfy

$$g(2x) + g(2y) = 2g(x+y)g(x-y).$$

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