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Solution of an Inverse Thermoelastic Problem of Heat Conduction with Internal Heat Generation in an Annular Disc

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Abstract

This paper consist of the inverse thermoelastic problem of heat conduction with internal heat generation for the determination of unknown temperature, displacement, stress function by using Marchi-Zgrablich transform and Finite Marchi-Fasulo integral transform. The results are obtained in the form of infinite series.

Keywords: Inverse thermoelastic problem, Temperature distribution, Stress functions.

1 Introduction

Nowacki, W. [3] investigated the state of stress in a thick circular plate due to a temperature field. Roychaudhari, S.K. [4] studied a note on the quasi-static thermal stresses in a thin circular plate due to transient temperature applied along the circumference of a circle over the upper face and a note on quasi-static thermal deflection of a thin clamped circular plate due to ramp-type heating of a concentric circular region of the upper face. Deshmukh, K. C. and Khobragade N. L. [1] have studied an inverse quasi-static thermal deflection problem for a thin clamped circular plate Wankhede, P.C.[5] have investigated on the quasi-static thermal stresses in a circular plate. Khobragade N. W. and Lamba N. K.[2] discussed the thermal stresses of a thin annular disc due to partially distributed heat supply. In this paper we consider the three dimensional problem of heat conduction and determine the temperature, displacement, stress function of annular disc by applying the Marchi-Fasulo integral transform and Marchi-Zgrablich transform.

2 Statement of the Problem

Consider a thin annular isotropic disc of thickness 2h occupying the space D:$a \le r \le b, -h \le z \le h$. The differential equation governing the displacement function $U(r,z)$ as

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} = (1+\nu)a_t T \quad (1)$$

With $U_r = 0$ at $r = a$ and $r = b$, ν and a_t are the Poisson's ratio and the linear coefficient of the thermal expansion of the material of the disc respectively and $T(r,z,t)$ is the temperature of the disc satisfying the differential equation

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} + \frac{g(r,z,t)}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \quad (2)$$

Where k and α thermal conductivity and thermal diffusivity of the material of the disc subject to the initial conditions

$$T(r,z,0) = T_0 \quad (3)$$

The boundary conditions and interior condition are

$$\left[T + k_1 \frac{\partial T(r,z,t)}{\partial r}\right]_{r=\xi} = f(\xi,z,t) \text{ (Known)} \quad (4)$$

$$\left[T + k_2 \frac{\partial T(r,z,t)}{\partial r}\right]_{r=a} = F_1\ (z,t) \quad (5)$$

$$\left[T + k_3 \frac{\partial T(r,z,t)}{\partial z}\right]_{z=h} = F_2\ (r,t) \quad (6)$$

$$\left[T + k_4 \frac{\partial T(r,z,t)}{\partial z}\right]_{z=-h} = F_3\ (r,t) \quad (7)$$

$$[T(r,z,t)]_{r=b} = G(z,t) \text{ (Unknown)} \quad (8)$$

The stress functions σ_{rr} and $\sigma_{\theta\theta}$ are given by

$$\sigma_{rr} = -2\mu \frac{1}{r}\frac{\partial U}{\partial r} \quad (9)$$

$$\sigma_{\theta\theta} = -2\mu \frac{\partial^2 U}{\partial r^2} \quad (10)$$

Where μ is the Lame's constant, while each of stress function $\sigma_{rz,}, \sigma_{zz}, \sigma_{\theta z}$ are zero within the disc in the plane state of stress.

The equations (1) to (10) constitute the mathematical formulation of the problem under consideration.

3 Solution of the problem

The finite Marchi-Zgrablich integral transform of order p is defined as

$$\bar{f}_p(n) = \int_a^\xi x f(x) S_p(\mathrm{k}_1, \mathrm{k}_2, \mu_n x) dx \quad (11)$$

And inverse Marchi-Zgrablich integral transform as

$$f(x) = \sum_{n=1}^{\infty} \frac{\bar{f}_p(n) S_p(\mathrm{k}_1, \mathrm{k}_2, \mu_n x)}{c_n} \quad (12)$$

Where

$S_p(\mathrm{k}_1, \mathrm{k}_2, \mu_n x) = J_p(\mu_n x)\{G_p(\mathrm{k}_1, \mu_n a) + G_p(\mathrm{k}_2, \mu_n \xi)\} - G_p(\mu_n x)\{J_p(\mathrm{k}_1, \mu_n a) + J_p(\mathrm{k}_2, \mu_n \xi)\}$

$c_n = \frac{\xi^2}{2}\{S_p{}^2(\mathrm{k}_1, \mathrm{k}_2, \mu_n \xi) - S_{p-1}(\mathrm{k}_1, \mathrm{k}_2, \mu_n \xi).S_{p+1}(\mathrm{k}_1, \mathrm{k}_2, \mu_n \xi)\} - \frac{a^2}{2}\{S_p{}^2(\mathrm{k}_1, \mathrm{k}_2, \mu_n a) - S_{p-1}(\mathrm{k}_1, \mathrm{k}_2, \mu_n a).S_{p+1}(\mathrm{k}_1, \mathrm{k}_2, \mu_n a)\}$

An operational property is given by

$$\int_a^\xi \left[\frac{\partial^2 f}{\partial x^2} + \frac{1}{x}\frac{\partial f}{\partial x} + \frac{p^2 f}{x^2}\right] S_p{}^2(\mathrm{k}_1, \mathrm{k}_2, \mu_n x) = \frac{\xi}{\mathrm{k}_2} S_p{}^2(\mathrm{k}_1, \mathrm{k}_2, \mu_n \xi)\left[f + \mathrm{k}_2 \frac{\partial f}{\partial x}\right]_{x=\xi} - \frac{a}{\mathrm{k}_1} S_p{}^2(\mathrm{k}_1, \mathrm{k}_2, \mu_n a)\left[f + \mathrm{k}_1 \frac{\partial f}{\partial x}\right]_{x=a} - \mu_n{}^2 \bar{f}_p(n) \quad (13)$$

The finite Marchi-Fasulo integral transform of f(z),-h < z < h is defined to be

$$\bar{F} = \int_{-h}^{h} f(z)\, P_n(z) dz \quad (14)$$

Then at each point of (-h,h) at which f(z) is continuous.Also the inverse finite Marchi-Fasulo transform is defined as

$$f(z) = \sum_{n=1}^{\infty} \frac{\bar{F}(n)}{\lambda_n} P_n(z) \quad (15)$$

Where

$P_n(z) = Q_n \cos(a_n z) - W_n \sin(a_n z)$

$Q_n = a_n(\alpha_1 + \alpha_2)\cos(a_n h) + (\beta_1 - \beta_2)\sin(a_n h)$

$W_n = (\beta_1 + \beta_2)\cos(a_n h) + (\alpha_1 - \alpha_2) a_n \sin(a_n h)$

$\lambda_n = \int_{-h}^{h} P_n{}^2(z) dz = h[Q_n{}^2 + W_n{}^2] + \frac{\sin(2a_n h)}{2a_n}[Q_n{}^2 - W_n{}^2]$

The eigen values a_n are the solutions of the equation

$$[\alpha_1 a\cos(ah) + \beta_1 \sin(ah)] \times [\beta_2 \cos(ah) + \alpha_2 a\sin(ah)] = [\alpha_2 a\cos(ah) - \beta_2 \sin(ah)]\,[\beta_1 \cos(ah) - \alpha_1 a\sin(ah)] \quad (16)$$

Where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are constants.

Applying finite Marchi-Zgrablich integral transform again finite Marchi-Fasulo transform and then their inverses stated in (11) to (16), to equations (2) to (8) ones obtain

$$T = \alpha \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{P_m(z)}{\lambda_m} e^{-(\mu_n{}^2+a_m)t} \times \left\{ \int e^{(\mu_n{}^2+a_m)t} \left[\frac{a}{k_2} S_0{}^2(k_1, k_2, \mu_n a)F_1{}^* - \frac{\xi}{k_1} S_0{}^2(k_1, k_2, \mu_n \xi)f^* + \frac{P_m(h)}{\beta_1} F_2 - \frac{P_m(-h)}{\beta_2} F_3 + \frac{\bar{g}^*}{K}\right] dt + \frac{T_1}{\alpha}\right\} \frac{S_0(k_1,k_2,\mu_n r)}{c_n} \quad (17)$$

$$G = \alpha \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{P_m(z)}{\lambda_m} e^{-(\mu_n{}^2+a_m)t} \times \left\{ \int e^{(\mu_n{}^2+a_m)t} \left[\frac{a}{k_2} S_0{}^2(k_1, k_2, \mu_n a)F_1{}^* - \frac{\xi}{k_1} S_0{}^2(k_1, k_2, \mu_n \xi)f^* + \frac{P_m(h)}{\beta_1} F_2 - \frac{P_m(-h)}{\beta_2} F_3 + \frac{\bar{g}^*}{K}\right] dt + \frac{T_1}{\alpha}\right\} \frac{S_0(k_1,k_2,\mu_n b)}{c_n} \quad (18)$$

Where

$$T_1 = T_0 - \left\{\alpha \int e^{(\mu_n{}^2+a_m)t} \left[\frac{a}{k_2} S_0{}^2(k_1, k_2, \mu_n a)F_1{}^* - \frac{\xi}{k_1} S_0{}^2(k_1, k_2, \mu_n \xi)f^* + \frac{P_m(h)}{\beta_1} F_2 - \frac{P_m(-h)}{\beta_2} F_3 + \frac{\bar{g}^*}{K}\right] dt\right\}_{t=0}$$

Also $\bar{g}^*$ denotes the finite Marchi-Fasulo transform of $\bar{g}$ and $\bar{g}$ denotes the finite Marchi-Zgrablich transform of$'g'$.

3 Determination of Thermoelastic displacement

Substituting the value of T from (17) in (1) it gets,

$$U(r, z, t) = -\alpha(1+\nu)a_t \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{P_m(z)}{\lambda_m} e^{-(\mu_n{}^2+a_m)t} \times \left\{ \int e^{(\mu_n{}^2+a_m)t} \left[\frac{a}{k_2} S_0{}^2(k_1, k_2, \mu_n a)F_1{}^* - \frac{\xi}{k_1} S_0{}^2(k_1, k_2, \mu_n \xi)f^* + \frac{P_m(h)}{\beta_1} F_2 - \frac{P_m(-h)}{\beta_2} F_3 + \frac{\bar{g}^*}{K}\right] dt + \frac{T_1}{\alpha}\right\} \frac{S_0(k_1,k_2,\mu_n r)}{c_n} \quad (19)$$

4 Determination of Stress functions

Using (19) in (9) and (10) the stress functions are obtained as

$$\sigma_{rr} = \frac{2\mu(1+\nu)a_t\alpha}{r} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{P_m(z)}{\lambda_m} e^{-(\mu_n{}^2+a_m)t} \times \left\{\left[\gamma + \frac{T_1}{\alpha}\right]' \frac{S_0(k_1,k_2,\mu_n r)}{c_n} + \left[\gamma + \frac{T_1}{\alpha}\right] \frac{S_0{}'(k_1,k_2,\mu_n r)}{c_n}\right\} \quad (20)$$

$$\sigma_{\theta\theta} = 2\mu(1+\nu)a_t\alpha \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{P_m(z)}{\lambda_m} e^{-(\mu_n{}^2+a_m)t} \times \left\{\left[\gamma + \frac{T_1}{\alpha}\right]'' \frac{S_0(k_1,k_2,\mu_n r)}{c_n} + \left[\gamma + \frac{T_1}{\alpha}\right] \frac{S_0{}''(k_1,k_2,\mu_n r)}{c_n}\right\} \quad (21)$$

Where

$$\gamma = \int e^{(\mu_n{}^2+a_m)t} \left[\frac{a}{k_2} S_0{}^2(k_1, k_2, \mu_n a)F_1{}^* - \frac{\xi}{k_1} S_0{}^2(k_1, k_2, \mu_n \xi)f^* + \frac{P_m(h)}{\beta_1} F_2 - \frac{P_m(-h)}{\beta_2} F_3 + \frac{\bar{g}^*}{K}\right] dt$$

5 Special Case

$$f(z,t) = (k_1 + \xi)(z-h)^2(z+h)^2 e^{-t} \tag{22}$$

$$g(r,z,t) = \delta(r - r_1) \tag{23}$$

Where δ is Dirac delta function.

Applying finite Marchi-Fasulo transform to (22) and finite Marchi-Zgrablich first, then finite Marchi-Fasulo to (23) it gets,

$$f^* = 4(k_3 + k_4)(k_1 + \xi)e^{-t} \times \left[\frac{(a_m h)cos^2(a_m h) - \cos(a_m h)sin(a_m h)}{a_m{}^2}\right] \tag{24}$$

$$\bar{g}^* = A \tag{25}$$

Substituting values from (24) and (25) in (17) to (21) , it obtains

$$T = \alpha \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{P_m(z)}{\lambda_m} e^{-(\mu_n{}^2 + a_m)t} \times \left\{ \int e^{(\mu_n{}^2+a_m)t} \left[\frac{a}{k_2} S_0{}^2(k_1, k_2, \mu_n a) 4(k_3 + k_4)(k_2 + b)e^{-t} \left[\frac{(a_m h)cos^2(a_m h) - \cos(a_m h)sin(a_m h)}{a_m{}^2}\right] - \frac{\xi}{k_1} S_0{}^2(k_1, k_2, \mu_n \xi) 4(k_3 + k_4)(k_1 + \xi)e^{-t} \times \left[\frac{(a_m h)cos^2(a_m h) - \cos(a_m h)sin(a_m h)}{a_m{}^2}\right] + \frac{P_m(h)}{\beta_1}F_2 - \frac{P_m(-h)}{\beta_2}F_3 + \frac{A}{K}\right] dt + \frac{T_1}{\alpha}\right\} \frac{S_0(k_1,k_2,\mu_n r)}{c_n}$$

$$G = \alpha \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{P_m(z)}{\lambda_m} e^{-(\mu_n{}^2 + a_m)t} \times \left\{ \int e^{(\mu_n{}^2+a_m)t} \left[\frac{a}{k_2} S_0{}^2(k_1, k_2, \mu_n a) 4(k_3 + k_4)(k_2 + b)e^{-t} \left[\frac{(a_m h)cos^2(a_m h) - \cos(a_m h)sin(a_m h)}{a_m{}^2}\right] - \frac{\xi}{k_1} S_0{}^2(k_1, k_2, \mu_n \xi) 4(k_3 + k_4)(k_1 + \xi)e^{-t} \times \left[\frac{(a_m h)cos^2(a_m h) - \cos(a_m h)sin(a_m h)}{a_m{}^2}\right] + \frac{P_m(h)}{\beta_1}F_2 - \frac{P_m(-h)}{\beta_2}F_3 + \frac{A}{K}\right] dt + \frac{T_1}{\alpha}\right\} \frac{S_0(k_1,k_2,\mu_n b)}{c_n}$$

Thermoelastic displacement is given by

$$U(r,z,t) = -\alpha(1+\nu)a_t \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{P_m(z)}{\lambda_m} e^{-(\mu_n{}^2 + a_m)t} \times \left\{ \int e^{(\mu_n{}^2+a_m)t} \left[\frac{a}{k_2} S_0{}^2(k_1, k_2, \mu_n a) 4(k_3 + k_4)(k_2 + b)e^{-t} \left[\frac{(a_m h)cos^2(a_m h) - \cos(a_m h)sin(a_m h)}{a_m{}^2}\right] - \frac{\xi}{k_1} S_0{}^2(k_1, k_2, \mu_n \xi) 4(k_3 + k_4)(k_1 + \xi)e^{-t} \times \left[\frac{(a_m h)cos^2(a_m h) - \cos(a_m h)sin(a_m h)}{a_m{}^2}\right] + \frac{P_m(h)}{\beta_1}F_2 - \frac{P_m(-h)}{\beta_2}F_3 + \frac{A}{K}\right] dt + \frac{T_1}{\alpha}\right\} \frac{S_0(k_1,k_2,\mu_n r)}{c_n}$$

Stress functions are given by

$$\sigma_{rr} = \frac{2\mu(1+\nu)a_t\alpha}{r} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{P_m(z)}{\lambda_m} e^{-(\mu_n{}^2 + a_m)t} \times \left\{ \left\langle \int e^{(\mu_n{}^2+a_m)t} \left[\frac{a}{k_2} S_0{}^2(k_1, k_2, \mu_n a) F_1{}^* - \frac{\xi}{k_1} S_0{}^2(k_1, k_2, \mu_n \xi) f^* + \frac{P_m(h)}{\beta_1}F_2 - \frac{P_m(-h)}{\beta_2}F_3 + \frac{\bar{g}^*}{K}\right] dt + \frac{T_1}{\alpha}\right\rangle' \frac{S_0(k_1,k_2,\mu_n r)}{c_n} + \left[\int e^{(\mu_n{}^2+a_m)t} \left[\frac{a}{k_2} S_0{}^2(k_1, k_2, \mu_n a) F_1{}^* - \frac{\xi}{k_1} S_0{}^2(k_1, k_2, \mu_n \xi) f^* + \frac{P_m(h)}{\beta_1}F_2 - \right.\right.$$

$$\frac{P_m(-h)}{\beta_2}F_3 + \frac{\bar{g}^*}{K}\Big] dt + \frac{T_1}{\alpha}\Big]\frac{S_0{}'(\mathrm{k}_1,\mathrm{k}_2,\mu_n r)}{c_n}\Big\}$$

$$\sigma_{\theta\theta} = 2\mu(1+\nu)a_t\alpha\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{P_m(z)}{\lambda_m}e^{-(\mu_n{}^2+a_m)t} \times \Big\{\langle\int e^{(\mu_n{}^2+a_m)t}\Big[\frac{a}{\mathrm{k}_2}S_0{}^2(\mathrm{k}_1,\mathrm{k}_2,\mu_n a)\mathrm{F}_1{}^* - \frac{\xi}{\mathrm{k}_1}S_0{}^2(\mathrm{k}_1,\mathrm{k}_2,\mu_n\xi)\mathrm{f}^* + \frac{P_m(h)}{\beta_1}F_2 - \frac{P_m(-h)}{\beta_2}F_3 + \frac{\bar{g}^*}{K}\Big] dt + \frac{T_1}{\alpha}\rangle''\frac{S_0(\mathrm{k}_1,\mathrm{k}_2,\mu_n r)}{c_n} + \Big[\int e^{(\mu_n{}^2+a_m)t}\Big[\frac{a}{\mathrm{k}_2}S_0{}^2(\mathrm{k}_1,\mathrm{k}_2,\mu_n a)\mathrm{F}_1{}^* - \frac{\xi}{\mathrm{k}_1}S_0{}^2(\mathrm{k}_1,\mathrm{k}_2,\mu_n\xi)\mathrm{f}^* + \frac{P_m(h)}{\beta_1}F_2 - \frac{P_m(-h)}{\beta_2}F_3 + \frac{\bar{g}^*}{K}\Big] dt + \frac{T_1}{\alpha}\Big]\frac{S_0{}''(\mathrm{k}_1,\mathrm{k}_2,\mu_n r)}{c_n}\Big\}$$

Where

$$f^* = 4(\mathrm{k}_3+\mathrm{k}_4)(\mathrm{k}_1+\xi)e^{-t} \times \left[\frac{(a_m h)cos^2(a_m h) - \cos(a_m h)sin(a_m h)}{a_m{}^2}\right]$$

And

$$\mathrm{F}_1{}^* = 4(\mathrm{k}_3+\mathrm{k}_4)(\mathrm{k}_2+b)e^{-t} \times \left[\frac{(a_m h)cos^2(a_m h) - \cos(a_m h)sin(a_m h)}{a_m{}^2}\right]$$

6 Numerical results

Thermal expansion coefficient	8.4×10^{-6}
Thermal conductivity	0.041
Thermal diffusivity	0.0733
Poisson ratio	0.34
Specific heat	0.124
Young's Modulus	116
Density	4.506
a	1
b	2
h	0.2
k_1, k_2	0.25
ξ	1.5
A	0.1
T_1	1
t	1

$$G = (8.4 \times 10^{-6})\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{P_m(z)}{\lambda_m}e^{-(\mu_n{}^2+a_m)} \times \Big\{\int e^{(\mu_n{}^2+a_m)}\Big[18 \times S_0{}^2(0.25, 0.25, \mu_n)e^{-1} \times$$

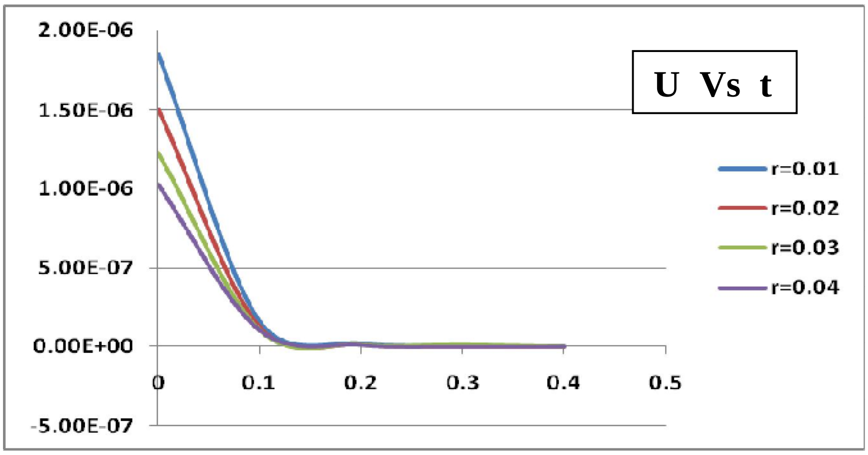

Fig-1: Variation of “U” and “t “for various values of “r”.

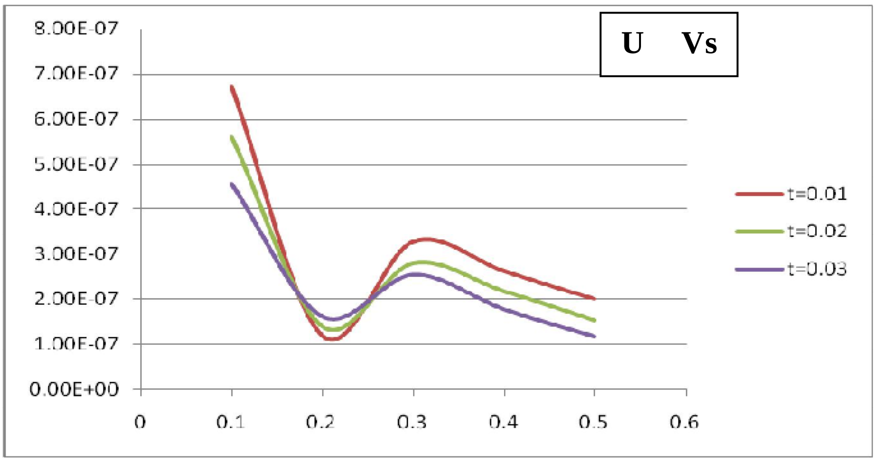

Fig-2: Variation of “U” and “r “for various values of “t”.

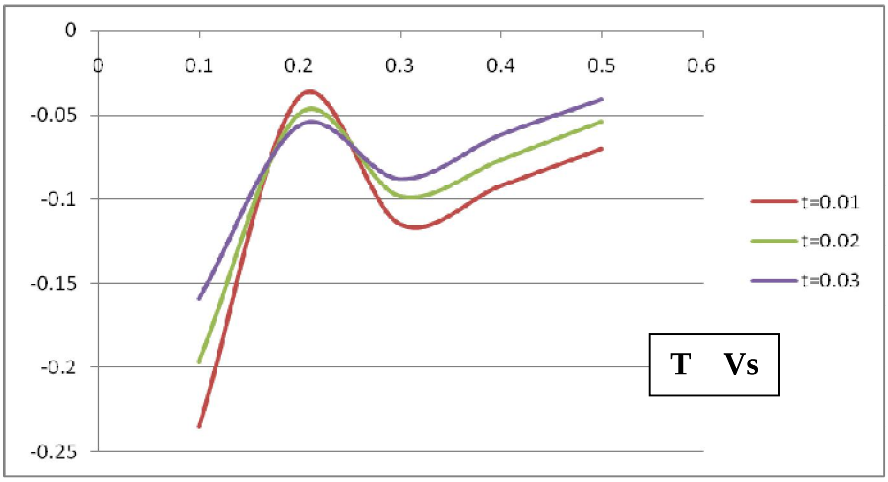

Fig-3: Variation of “T” and “r “for various values of “t”.

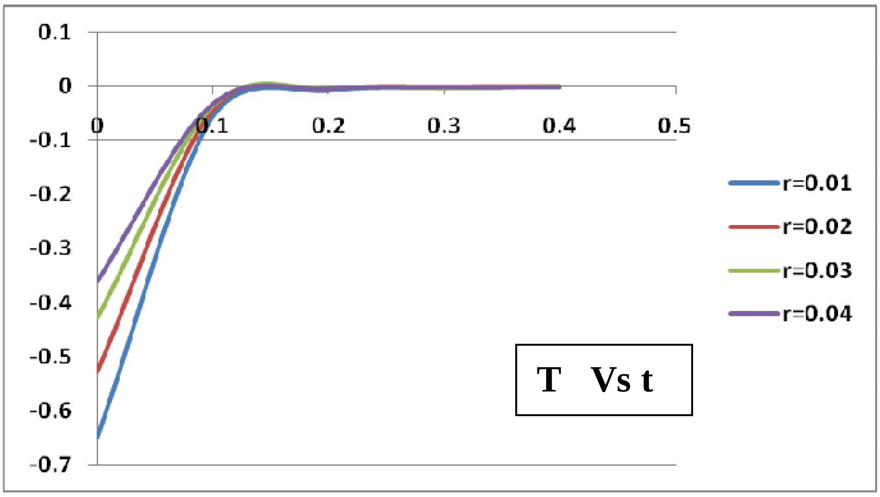

Fig-4: Variation of “T” and “t “for various values of “r”.

6 Conclusions

In this paper temperature distribution, thermal stresses have been investigated. The temperature distribution and thermal stresses have been determined by using Marchi-Zgrablich and finite Marchi-Fasulo transform. The results are obtained in the form of Bessel functions and infinite series.

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