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Bounds for the Zeros of a Complex Polynomial with Restricted Coefficients

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Abstract

If P(z) be a polynomial of degree n with decreasing coefficients, then all its zeros lie in $|z| \leq 1$. In this paper we find new classical results concerning Enestrom-Kakeya theorem and related analytic functions. Besides several consequences, our results improve the bounds by relaxing and weakening the hypothesis in some cases.

Keywords: Polynomials, Enestrom - Kakeya theorem ,The sharper bounds.

1.Introduction:

The following result due to Enestrom and Kakeya [6] is well known in the theory of distribution of zeros of polynomials.

Theorem 1 : If $P(z) = \sum_0^n a_j z^j$ be a polynomial of degree n such that

$a_n \geq a_{n-1} \geq a_{n-2} \geq\text{------}\geq a_1 \geq a_0 > 0$, $a_j \in R$ (1)

Then P(z) has all its zeros in $|z|\leq 1$

Joyal et al [11] extended theorem to the polynomials whose coefficient are monotonic but not necessarily non negative and proved the following:

Theorem 2 : If $P(z) = \sum_0^n a_j z^j$ be a polynomial of degree n such that

$a_n \geq a_{n-1} \geq a_{n-2} \geq\text{------}\geq a_1 \geq a_0$, $a_j \in R$

Then all the zeros of P(z) lie in

$|z| \leq (a_n - a_0 + |a_0|) \div |a_n|$. (2)

Theorem 3 : If $P(z) = \sum_0^n a_j z^j$ be a polynomial of degree n such that for some $\lambda \geq 1$,

$$\lambda a_n \geq a_{n-1} \geq a_{n-2} \geq \text{------} \geq a_1 \geq a_0 ,\ \lambda, a_j \in R, \quad (3)$$

then all the zeros of P(z) lie in

$$|z + \lambda - 1| \leq (\lambda a_n - a_0 + |a_0|) \div |a_n|. \quad (4)$$

Among other authors besides Joyal et al[11], Dewan and Govil[6] and Aziz and Zarger[1] also extended Theorem 1 to the polynomials whose coefficients are monotonic but not necessarily non negative.

2.The polynomials with complex coefficients:

Govil and Mc Tune[10] extended the results of Aziz and Zarger[1] to the polynomials with complex coefficients given by:

Theorem 4 : Let $P(z) = \sum_0^n a_j z^j$ be a polynomial of degree n with $Re(a_j) = \alpha_j$ and $Im(a_j) = \beta_j$, for j = 0,1,2-----n. If for some $\lambda \geq 1$,

$$\lambda \alpha_n \geq \alpha_{n-1} \geq \alpha_{n-2} \geq \text{------} \geq \alpha_1 \geq \alpha_0 ,\ \lambda, a_j \in R, \quad (5)$$

then all the zeros of P(z) lie in

$$|z + \lambda - 1| \leq (\lambda \alpha_n - \alpha_0 + |\alpha_0| + 2\sum_0^n |\beta_j|) \div |a_n| \quad (6)$$

In this paper we discuss certain properties given by the following:

Theorem 5 : Let $P(z) = \sum_0^n a_j z^j$ be a polynomial of degree n with complex co-efficents such that $Re(a_j) = \alpha_j$ and $Im(a_j) = \beta_j$, for j= 0,1,2-----n. and If for some $\lambda, \mu \geq 1$, $0 < \gamma, \delta \leq 1$

$$\lambda \alpha_n \geq \alpha_{n-1} \geq \alpha_{n-2} \geq \text{------} \geq \alpha_1 \geq \gamma \alpha_0$$
$$\mu \beta_n \geq \beta_{n-1} \geq \beta_{n-2} \geq \text{------} \geq \beta_1 \geq \delta \beta_0 \quad (7)$$

then all the zeros of P(z) lie in the disc:

$$\left|z + \frac{(\lambda-1)\alpha_n + i(\mu-1)\beta_n}{a_n}\right|$$
$$\leq \frac{1}{|a_n|}[\{(\lambda\alpha_n - \alpha_{n-1})^2 + (\mu\beta_n - \beta_{n-1})^2\}^{1/2} + 2(\gamma|\alpha_0| + \delta|\beta_0|)$$
$$- (\gamma|\alpha_0| - \delta|\beta_0|)] \quad (8)$$

Proof: Consider the polynomial

$F(z) = (1-z)P(z) = (1-z)(a_0 + a_1z + a_2z^2 + a_3z^3 + \ldots + a_{n-1}z^{n-1} + a_nz^n)$

Let $|z| > 1$. Then

$$F(z)| \geq |-a_nz^{n+1} - (\lambda\alpha_n - \alpha_n)z^n + (\lambda\alpha_n - \alpha_{n-1})z^n - i(\mu\beta_n - \beta_n)z^n + i(\mu\beta_n - \beta_{n-1})z^n + \sum_{j=1}^{j=n-1}(\alpha_j - \alpha_{j-1})z^j \quad + i\sum_{j=1}^{j=n-1}(\beta_j - \beta_{j-1})z^j + (\alpha_1 - \gamma\alpha_0)z + (\gamma\alpha_0 - \alpha_0)z + i(\beta_1 - \delta\beta_0)z + i(\delta\beta_0 - \beta_0)z + a_0$$

Therefore, $|F(z)| \leq [|F_1(\lambda,\mu,\alpha,\beta,z)| - |F_2(\lambda,\mu,\alpha,\beta)| + |F_3(\alpha,z)| + |F_4(\beta,z)|$ (9)

Hence

$|F_1(\lambda,\mu,\alpha,\beta,z)| = |\, a_n z+(\lambda-1)\alpha_n + i(\mu-1)\beta_n \,|$ (10)

Now, the lemma due to Govil & Rehman[9] is given as:

Lemma: If $|\arg a_j - \beta| \le \alpha \le \pi/2$ for some t>0 , $|ta_j| \ge |a_{j-1}|$, then

$|\, ta_j - a_{j-1}| \le \{(|ta_j|-|a_{j-1}|)\cos\alpha + (|ta_j|+|a_{j-1}|)\sin\alpha\}$ (11)

Hence

$|F_2(\lambda,\mu,\alpha,\beta)\,| = [\{\, (\lambda\alpha_n - \alpha_{n-1})^2 + (\mu\beta_n - \beta_{n-1})^2\}^{1/2}]$ (12)

Also

$|F_3(\alpha,z) \quad | \le \alpha_{n-1} + |(\gamma\alpha_0 - \gamma\alpha_0 + \alpha_0)z^{-n}|.$

$\le \alpha_{n-1} + \gamma|\alpha_0| + [(|\gamma\alpha_0| - |\alpha_0|)\cos\alpha + (|\gamma\alpha_0| + |\alpha_0|)\sin\alpha], \gamma > 0$ (13)

Similarly

$|\, F_4(\alpha,z)| \le |\beta_{n-1}| + \delta|\beta_0| + [\big((\delta\beta_0) - |\beta_0|\big)cos\alpha + \big((\delta\beta_0) + |\beta_0|\big)sin\alpha]$ (14)

Therefore, from eq.(9), taking into the account of the result of the equations (10),(12),(13) and (14), we have

$|F(z)| \le \ |\ a_n z+(\lambda-1)\alpha_n \ + i(\mu-1)\beta_n \ | \ + [\{\, (\lambda\alpha_n - \alpha_{n-1})^2 + (\mu\beta_n - \beta_{n-1})^2\}^{1/2} + \alpha_{n-1} + \gamma|\alpha_0| - [(|\gamma\alpha_0| - |\alpha_0|)\cos\alpha + (|\gamma\alpha_0| + |\alpha_0|)\sin\alpha] + |\beta_{n-1}| + \delta|\beta_0| + [\big((\delta\beta_0) - |\beta_0|\big)cos\alpha + \big((\delta\beta_0) + |\beta_0|\big)sin\alpha]$ (15)

Thus for $|z|>1$, $|F(z)|>0$ only if

$|a_n z+(\lambda-1)\alpha_n + i(\mu-1)\beta_n| > [\{\, (\lambda\alpha_n - \alpha_{n-1})^2 + (\mu\beta_n - \beta_{n-1})^2\}^{1/2} + \alpha_{n-1} + \gamma|\alpha_0| - [(|\gamma\alpha_0| - |\alpha_0|)\cos\alpha + (|\gamma\alpha_0| + |\alpha_0|)\sin\alpha] + |\beta_{n-1}| + \delta|\beta_0| + [\big((\delta\beta_0) - |\beta_0|\big)cos\alpha + \big((\delta\beta_0) + |\beta_0|\big)sin\alpha]$ (16)

which gives

$|z+\frac{(\lambda-1)\alpha_n + i(\mu-1)\beta_n}{\alpha_n}| > \frac{1}{|a_n|}[|(\{\, (\lambda\alpha_n - \alpha_{n-1}) + i(\mu\beta_n - \beta_{n-1})| + \alpha_{n-1} + \gamma(1 + cos\alpha + \sin\alpha)|\alpha_0| + (\sin\alpha - cos\alpha)|\alpha_0| + \beta_{n-1} + \delta(1 + cos\alpha + \sin\alpha)|\beta_0| + (\sin\alpha - cos\alpha)|\beta_0|\,]$ (17)

Above equation shows that the zeros of F(z) having modulii greater than 1 lie in the circle

$|z+\frac{(\lambda-1)\alpha_n + i(\mu-1)\beta_n}{\alpha_n}| \le \frac{1}{|a_n|}[\{\, (\lambda\alpha_n - \alpha_{n-1})^2$

$+(\mu\beta_n - \beta_{n-1})^2\}^{1/2} + 2\,(\gamma|\alpha_0| + \delta|\beta_0|) - (\gamma|\alpha_0| - \delta|\beta_0|)]$ (18)

It can also be verified that the zeros of F(z) whose modulus is less than or equal to one also lie in the circle defined by equation(ii) and therefore all the zeros of P(z) lying in the disc given by equation(ii).Now, when $\alpha = 0$, then L.H.S. becomes

$$|z+\frac{(\lambda-1)\alpha_n+i(\mu-1)\beta_n}{a_n}|$$
$$=\frac{1}{|a_n|}\left[\{(\lambda\alpha_n-\alpha_{n-1})^2+(\mu\beta_n-\beta_{n-1})^2\}^{1/2}+(\alpha_{n-1}+\beta_{n-1})+|\alpha_0|+|\beta_0|\right]$$

which proves Th. 5.

Illustration: Now we give some examples to show that the present estimate given by our Th. 5 are sharper as compared to the other authors. We therefore construct a polynomial P(z) $=\sum_0^n a_j\, z^j$ correspoding to n=2, 3 & 4 and compare the bounds obtained by other authors with our present bounds and thereby give the location of zeros of the polynomials corresponding to these values of n.

N	$a_j=\alpha_j+i\beta_j$	Approximate zeros of polynomials $P_n(z)$	Different values of λ, μ, δ and γ	Bounds for the zeros of the polynomials by the present estimate	Comparison of present estimate with other authors
2	a_2=(2,3), a_1=(-2,2), a_0 =(-5,5) with constraint $\lambda\alpha_2 \geq \alpha_1 \geq \gamma\alpha_0$ and $\mu\beta_2 \geq \beta_1 \geq \delta\beta_0$	z_1= 3.17-0.905i z_2 = 2.5 + 0.75i	Case-(i) λ = 3, μ = 3, $\delta=1$, $\gamma=1$	|z |≤ 5.47 from Th.5	|z |≤ 10.986 (without any constr -aint on β_i's) from Th.2
					|z |≤ 11.096 (without any constraint on β_i's) from Th.3
					|z |≤ 10.896 from Th.4
			Case-(ii) λ = 3, μ = 2, $\delta=1$, $\gamma=1$	|z |≤ 6.12 from Th. 5	|z |≤ 10.986 (even without any constraint on β_i's) from Th.2
					|z |≤ 11.096 (even without any constraint on β_i's) from Th.3.

n	$a_j = \alpha_j + i\beta_j$	Approximate zeros of polynomials $P_n(z)$	Different values of λ, μ, δ and γ	Bounds for the zeros of the polynomials by the present estimate	Comparison of present estimate with other authors
3	$a_3=2+3i$, $a_2=2+4i$, $a_1=-9-9i$ $a_0=-10-10i$ with constraint $\lambda\alpha_3 \geq \alpha_2 \geq \alpha_1 \geq \gamma\alpha_0$ $\mu\beta_3 \geq \beta_2 \geq \beta_1 \geq \delta\beta_0$	$z_1=-2+0i$ $z_2 = 3.17-0.905i$ $z_3 = 2.5+0.75i$	Case-(i) When λ = 3, μ = 3, $\delta = 1$, $\gamma = 1$	$\|z\| \leq 11$ from Th. 5	$\|z\| \leq 23.64$ from Th.2
			Case-(ii) When λ = 3, μ = 2, $\delta = 1$, $\gamma = 1$	$\|z\| \leq 8.4$ from Th. 5	$\|z\| \leq 23.64$ from Th.3
			Case(iii) When λ = 2, μ = 3, $\delta = 1$, $\gamma = 1$	$\|z\| \leq 9.3$ from Th. 5	$\|z\| \leq 22.08$ from Th.4

n	$a_j = \alpha_j + i\beta_j$	Approximate zeros of polynomials $P_n(z)$	Different values of λ, μ, δ and γ	Bounds for the zeros of the polynomials by the present estimate	Comparison of present estimate with other authors
4	$a_3=1+0i$, $a_2=0+0i$, $a_1=0+0i$ $a_0=-i$ with constraint $\lambda\alpha_3 \geq \alpha_2 \geq \alpha_1 \geq \gamma\alpha_0$ $\mu\beta_3 \geq \beta_2 \geq \beta_1 \geq \delta\beta_0$	$z_1= 0.9+0.4i$ $z_2 = 0.38+0.92i$ $z_3 = -0.38+0.92i$ $z_4 = -0.92+0.38i$	Case-(i) When λ = 3, μ = 3, $\delta = 1$, $\gamma = 1$	$\|z\| \leq 6.0$ from Th.5	$\|z\| \leq 7$ from Th.2
			Case-(ii) When λ = 2, μ = 3, $\delta = 1$, $\gamma = 1$	$\|z\| \leq 6.0$ from Th.5	$\|z\| \leq 7$ from Th.3
			Case-(iii) When λ = 2,	$\|z\| \leq 3.0$ from Th.5	$\|z\| \leq 5$ from Th.4

			$\mu = 3$, $\delta = 1$, $\gamma = 1$		

Remark: From the above table one can easily find that the present estimates are sharper for different values of λ, μ, δ *and* γ in all the cases.

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