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Oscillation Theorems for Certain Fourth Order Non-Linear Difference Equations

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Abstract

In this paper some sufficient conditions for the oscillation of all solutions of certain fourth order nonlinear difference equation of the form

$$\Delta^2\left(\frac{1}{p_n}\Delta^2 y_n\right)+q_n f\left(y_{n-\tau_n}\right)=0, n=0,1,2\ldots\ldots,$$

are obtained. Examples are given to illustrate the results.

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Introduction

In this paper, we are concerned with the oscillatory behavior of all the solutions of the nonlinear difference equation of the form

$$\Delta^2\left(\frac{1}{p_n}\Delta^2 y_n\right)+q_n f\left(y_{n-\tau_n}\right)=0, n=0,1,2,\ldots\ldots, \tag{1.1}$$

where Δ is the forward difference operator defined by $\Delta y_n = y_{n+1} - y_n$ and the following conditions are assumed to hold.

(H1) $\{q_n\}$ is a sequence of real numbers and $\{p_n\}$ is a sequence of positive numbers.

H2) $\{\tau_n\}$ is a sequence of integers such that $\lim_{n\to\infty}(n-\tau_n)=\infty.$

(H3) $R_n = \sum_{k=0}^{n-1} p_k \to \infty$ as $n \to \infty$.

(H4) $f : R \to R$ is continuous with $yf(y) > 0$, for $y \neq 0$.

By a solution of equation (1.1), we mean a real sequence $\{y_n\}$ which is defined for $n \geq \min_{i\geq 0}(i - \tau_i)$ and satisfies equation (1.1) for all large n. A solution $\{y_n\}$ is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise it is called non-oscillatory.

In recent years, much research is going in the study of oscillatory behavior of solutions of fourth order difference equations. For more details, on oscillatory behavior of difference equations, one may refer[1-29].

2 Main results

In this section, we present some sufficient conditions for the oscillation of all solutions of the equation (1.1).

Theorem 2.1 *Suppose that*

i) $q_n \geq 0$ and $\sum_{k=0}^{\infty} q_k = \infty$,

ii) $\liminf_{n\to\infty} f(y) > 0.$

Then every solution of equation (1.1) is oscillatory.

Proof: Suppose on the contrary that $\{y_n\}$ is non-oscillatory solution of equation (1.1). We assume that $\{y_n\}$ is eventually positive. Then there exists a positive integer n_0 such that

$$y_{n-\tau_n} > 0 \text{ for } n \geq n_0. \tag{2.1}$$

From equation (1.1), we have

$$\Delta^2\left(\frac{1}{p_n}\Delta^2 y_n\right) = -q_n f\left(y_n - \tau_n\right) \leq 0, n \geq n_0.$$

Then $\Delta\left\{\frac{1}{p_n}\Delta^2 y_n\right\}$ is an eventually non-increasing sequence.

We claim $\Delta\left(\frac{1}{p_n}\Delta^2 y_n\right) \geq 0$ for $n \geq n_0$.

We assume the contradiction that there is an $n_1 \geq n_0$ such that

$$\Delta\left(\frac{1}{p_n}\Delta^2 y_n\right) < 0, \qquad n \geq n_1.$$

That is,

$$\Delta\left(\frac{1}{p_n}\Delta^2 y_n\right) < -k_1, k_1 > 0, n_2 \geq n_1.$$

Summing the last inequality from n_2 to $(n-1)$, we have

$$\sum_{s=n_2}^{n-1}\Delta\left(\frac{1}{p_s\Delta^2 y_s}\right) \leq \sum_{s=n_2}^{n-1}(-k_1).$$

This implies

$$\frac{1}{p_n}\Delta^2 y_n - \frac{1}{p_{n_2}} \leq (-k_1)(n-n_2).$$

That is,

$$\frac{1}{p_n}\Delta^2 y_n \leq \frac{1}{p_{n_2}} + (-k_1)(n-n_2),$$

Therefore,

$$\frac{1}{p_n}\Delta^2 y_n \to -\infty.$$

Therefore, there exists an integer $n_3 \geq n_2$ such that

$$\frac{1}{p_n}\Delta^2 y_n < -k_2, k_2 > 0, n \geq n_3.$$

Summing the last inequality from n_3 to $(n-1)$, we obtain,

$$\Delta y_n - \Delta y_{n_3} \leq (-k_2)\sum_{s=n_3}^{n-1} p_s,$$

which implies

$$\Delta y_n \leq \Delta y_{n_3} - k_2\sum_{s=n_3}^{n-1} p_s.$$

That is

$$\Delta y_n \to -\infty \text{ as } n \to \infty.$$

Hence there is an integer $n_4 \geq n_3$ such that

$$\Delta^2 y_n < -k_3, k_3 > 0, \quad n \geq n_4.$$

Summing the last inequality from n_4 to $(n-1)$, we have

$$y_n - y_{n_4} \leq \sum_{s=n_4}^{n-1} (-k_3).$$

This implies

$$y_n \leq y_{n_4} - k_3(n - n_4).$$

Therefore

$$y_n \to -\infty \text{ as } n \to \infty.$$

which is a contradiction to the fact that $y_n > 0$ for $n \geq n_1$.

Hence

$$\Delta\left(\frac{1}{p_n}\Delta^2 y_n\right) \geq 0, n \geq n_0.$$

Therefore we have,

$y_{n-\tau_n} > 0$, $\Delta y_n \geq 0$, $\frac{1}{p_n}\Delta^2 y_n \geq 0$, $\Delta\left(\frac{1}{p_n}\Delta^2 y_n\right) \geq 0$ and $\Delta^2\left(\frac{1}{p_n}\Delta^2 y_n\right) \leq 0$ for $n \geq n_0$.

Let

$$L = \lim_{n\to\infty} y_n.$$

Then $L > 0$ is finite or infinite.

Case(i): $L > 0$ is finite.

Since f is a continuous function, we have

$$\lim_{n\to\infty} f(y_{n-\tau_n}) = f(L) > 0.$$

Then there exists a positive integer $n_5 \geq n_0$ such that

$$f(y_{n-\tau_n}) \geq \frac{1}{2} f(L), \; n \geq n_5. \tag{2.2}$$

By substituting inequality (2.2) in equation (1.1), we obtain

$$\Delta^2\left(\frac{1}{p_n}\Delta^2 y_n\right) + \frac{1}{2} f(L) q_n \leq 0, \; n \geq n_5. \tag{2.3}$$

Summing the last inequality from n_5 to n, we have

$$\Delta\left(\frac{1}{p_{n+1}}\Delta^2 y_{n+1}\right)-\Delta\left(\frac{1}{p_{n_5}}\Delta^2 y_{n_5}\right)+\frac{1}{2}f(L)\sum_{s=n_5}^{n} q_s \le 0,$$

which implies

$$\frac{1}{2}f(L)\sum_{s=n_5}^{n} q_s \le \Delta\left(\frac{1}{p_{n_5}}\Delta^2 y_{n_5}\right), n \ge n_5.$$

which contradicts (i).

Case(2): $L=\infty$
From the condition (ii), we have

$$\liminf_{n\to\infty} f\left(y_{n-\tau_n}\right) > 0.$$

So we may choose a positive constant c and a positive integer n_6 such that

$$f\left(y_{n-\tau_n}\right) \ge c, n \ge n_6. \tag{2.4}$$

Substituting inequality (2.4) in equation (1.1), we have

$$\Delta^2\left(\frac{1}{p_n}\Delta^2 y_n\right)+cq_n \le 0,\ n \ge n_6.$$

Summing the last inequality from n_6 to n, we get

$$c\sum_{s=n_6}^{n} q_s \le \Delta\left(\frac{1}{p_{n_6}}\Delta^2 y_{n_6}\right), n \ge n_6.$$

which contradicts (i).
This completes the proof.

Theorem 2.2 *Assume that* $q_n \ge 0$ *and* $\sum_{k=0}^{\infty}\frac{(k-N)}{R_k}q_k=\infty$

Then every bounded solution of equation (1.1) is oscillatory.

Proof: Proceeding as in the proof of Theorem 2.1 with assumption that equation (1.1) has a non-oscillatory bounded solution $\{y_n\}$, we get the inequality (2.3),

$$\Delta^2\left(\frac{1}{p_n}\Delta^2 y_n\right)+\frac{1}{2}f(L)q_n \le 0,\ n \ge n_5.$$

Dividing tha last inequality by R_n, we obtain

$$\frac{1}{R_n}\Delta^2\left(\frac{1}{p_n}\Delta^2 y_n\right)+\frac{1}{2R_n}f(L)q_n \le 0. \tag{2.5}$$

Consider

$$\sum_{s=N}^{n-1}\Delta^2\left(\frac{1}{p_s}\Delta^2 y_s\right)=\Delta\left(\frac{1}{p_n}\Delta^2 y_n\right)-\Delta\left(\frac{1}{p_N}\Delta^2 y_N\right),$$

which implies

$$(n-N)\Delta^2\left(\frac{1}{p_n}\Delta^2 y_n\right)=\Delta\left(\frac{1}{p_n}\Delta^2 y_n\right)-\Delta\left(\frac{1}{p_N}\Delta^2 y_N\right).$$

That is,

$$(n-N)\Delta^2\left(\frac{1}{p_n}\Delta^2 y_n\right)\le\Delta\left(\frac{1}{p_n}\Delta^2 y_n\right).$$

Then inequality (2.5) becomes,

$$\frac{\Delta\left(\frac{1}{p_n}\Delta^2 y_n\right)}{R_n(n-N)}+\frac{1}{2R_n}f(L)q_n \le 0.$$

That is,

$$\frac{1}{R_n}\Delta\left(\frac{1}{p_n}\Delta^2 y_n\right)+\frac{n-N}{2R_n}f(L)q_n \le 0. \tag{2.6}$$

We know that

$$\begin{aligned}\Delta\left(\frac{1}{R_n}\frac{1}{p_n}\Delta^2 y_n\right) &= \frac{1}{R_{n+1}}\Delta\left(\frac{1}{p_n}\Delta^2 y_n\right)+\frac{1}{p_n}\Delta^2 y_n\Delta\frac{1}{R_n} \\ &\le \frac{1}{R_n}\Delta\left(\frac{1}{p_n}\Delta^2 y_n\right)+\frac{1}{p_n}\Delta^2 y_n\Delta\frac{1}{R_n},\end{aligned}$$

which implies

$$\frac{1}{R_n}\Delta\left(\frac{1}{p_n}\Delta^2 y_n\right)\ge\Delta\left(\frac{1}{R_n}\frac{1}{p_n}\Delta^2 y_n\right)-\frac{1}{p_n}\Delta^2 y_n\Delta\frac{1}{R_n}. \tag{2.7}$$

Substituting (2.7) in (2.6), we have

$$\begin{aligned}\frac{n-N}{2R_n} f(L)q_n &\leq -\frac{1}{R_n}\Delta\left(\frac{1}{p_n}\Delta^2 y_n\right)\\ &\leq -\Delta\left(\frac{1}{R_n}\frac{1}{p_n}\Delta^2 y_n\right)+\frac{1}{p_n}\Delta^2 y_n \Delta\frac{1}{R_n}\\ &\leq -\Delta\left(\frac{1}{R_n}\frac{1}{p_n}\Delta^2 y_n\right)+\Delta^2 y_n\left(-\frac{1}{R_{n+1}^2}\right)\\ &\leq -\Delta\left(\frac{1}{R_n}\frac{1}{p_n}\Delta^2 y_n\right)+\Delta^2 y_n.\end{aligned}$$

That is,

$$\Delta\left(\frac{1}{R_n}\frac{1}{p_n}\Delta^2 y_n\right)-\Delta^2 y_n+\frac{n-N}{2R_n} f(L)q_n \leq 0.$$

Summing the last inequality from n_5 to n, we get

$$\sum_{s=n_5}^{n}\Delta\left(\frac{1}{R_s}\frac{1}{p_s}\Delta^2 y_s\right)-\sum_{s=n_5}^{n}\Delta^2 y_s+\frac{1}{2} f(L)\sum_{s=n_5}^{n}\frac{s-N}{R_s}q_s \leq 0.$$

This implies

$$\frac{1}{R_{n+1}}\frac{1}{p_{n+1}}\Delta^2 y_{n+1}-\frac{1}{R_{n_5}}\frac{1}{p_{n_5}}\Delta^2 y_{n_5}-\Delta y_{n+1}+\Delta y_{n_5}+\frac{1}{2} f(L)\sum_{s=n_5}^{n}\frac{s-N}{R_s}q_s \leq 0.$$

That is,

$$\begin{aligned}\frac{1}{2} f(L)\sum_{s=n_5}^{n}\frac{s-N}{R_s}q_s &\leq \frac{1}{R_{n_5}}\frac{1}{p_{n_5}}\Delta^2 y_{n_5}+\Delta y_{n+1}-\Delta y_{n_5}\\ &\leq \frac{1}{R_{n_5}}\frac{1}{p_{n_5}}\Delta^2 y_{n_5}+y_{n+2}-y_{n+1}-\Delta y_{n_5}.\end{aligned}$$

Since $\{y_n\}$ is bounded, we may choose a positive constant c such that

$$\sum_{s=n_5}^{n}\frac{s-N}{R_s}q_s \leq c,\ n\geq n_5, \tag{2.8}$$

which is a contradiction to the assumption of the theorem.

Theorem 2.3 *Assume that*

$(n-\tau_n)$ is nondecreasing, where $\tau_n \in \{0,1,2,....\}$ and $\sum_{n=0}^{\infty} q_n = \infty.$

Then the difference $\Delta\left(\frac{1}{p_n}\Delta^2 y_n\right)$ of every solution $\{y_n\}$ of equation (1.1) oscillates.

Proof: Assume the contradiction that equation (1.1) has a solution $\{y_n\}$ such that its difference $\Delta(\frac{1}{p_n}\Delta^2 y_n)$ is non-oscillatory. Assume first that the sequence $\Delta\left(\frac{1}{p_n}\Delta^2 y_n\right)$ is eventually negative. Then we may choose a positive integer n_0 such that $\Delta\left(\frac{1}{p_n}\Delta^2 y_n\right) < 0.$

Set

$$v_n = \frac{\Delta\left(\frac{1}{p_n}\Delta^2 y_n\right)}{f\left(y_{n-\tau_n}\right)},\ n \geq n_1 \geq n_0. \tag{2.9}$$

Then

$$\Delta v_n = \frac{f(y_n-\tau_n)\Delta^2\left(\frac{1}{p_n}\Delta^2 y_n\right) - \Delta\left(\frac{1}{p_n}\Delta^2 y_n\right)\Delta f\left(y_{n-\tau_n}\right)}{f\left(y_{n-\tau_n}\right) f\left(y_{n+1-\tau_{n+1}}\right)}. \tag{2.10}$$

Now,

$$\begin{aligned}
\Delta^2\left(\frac{1}{p_n}\Delta^2 y_n\right) &= \Delta\left(\frac{1}{p_{n+1}}\Delta^2 y_{n+1}\right) - \Delta\left(\frac{1}{p_n}\Delta^2 y_n\right) \\
&= \frac{1}{p_{n+2}}\Delta^2 y_{n+2} - \frac{1}{p_{n+1}}\Delta^2 y_{n+1} - \frac{1}{p_{n+1}}\Delta^2 y_{n+1} + \frac{1}{p_n}\Delta^2 y_n \\
&= \frac{1}{p_{n+2}}\Delta^2 y_{n+2} - \frac{2}{p_{n+1}}\Delta^2 y_{n+1} + \frac{1}{p_n}\Delta^2 y_n.
\end{aligned}$$

and

$$\Delta\left(\frac{1}{p_n}\Delta^2 y_n\right)\Delta f\left(y_{n-\tau_n}\right) = \left(\frac{1}{p_{n+1}}\Delta^2 y_{n+1} - \frac{1}{p_n}\Delta^2 y_n\right)\left(f\left(y_{n+1-\tau_{n+1}}\right) - f(y_{n-\tau_n})\right)$$
$$= \frac{1}{p_{n+1}}\Delta^2 y_{n+1} f\left(y_{n+1-\tau_{n+1}}\right) - \frac{1}{p_n}\Delta^2 y_n f\left(y_{n+1-\tau_{n+1}}\right)$$
$$- \frac{1}{p_{n+1}}\Delta^2 y_{n+1} f\left(y_{n-\tau_n}\right) + \frac{1}{p_n}\Delta^2 y_n \Delta f\left(y_{n-\tau_n}\right).$$

Therefore (2.10) becomes

$$\Delta v_n = \frac{\frac{1}{p_{n+2}}\Delta^2 y_{n+2}}{f\left(y_{n+1-\tau_{n+1}}\right)} - \frac{\frac{1}{p_{n+1}}\Delta^2 y_{n+1}}{f\left(y_{n+1-\tau_{n+1}}\right)} - \frac{\frac{1}{p_{n+1}}\Delta^2 y_{n+1}}{f\left(y_{n-\tau_n}\right)} + \frac{\frac{1}{p_n}\Delta^2 y_n}{f\left(y_{n-\tau_n}\right)}$$
$$= \frac{\Delta\left(\frac{1}{p_{n+1}}\Delta^2 y_{n+1}\right)}{f\left(y_{n+1-\tau_{n+1}}\right)} - \frac{\Delta\left(\frac{1}{p_n}\Delta^2 y_n\right)}{f\left(y_{n-\tau_n}\right)}$$
$$= \frac{\Delta\left(\frac{1}{p_{n+1}}\Delta^2 y_{n+1}\right)}{f\left(y_{n+1-\tau_{n+1}}\right)} - \frac{\left[\Delta\left(\frac{1}{p_{n+1}}\Delta^2 y_{n+1}\right) - \Delta^2\left(\frac{1}{p_n}\Delta^2 y_n\right)\right]}{f\left(y_{n-\tau_n}\right)}$$
$$= \frac{\Delta^2\left(\frac{1}{p_n}\Delta^2 y_n\right)}{f\left(y_{n-\tau_n}\right)} + \Delta\left(\frac{1}{p_{n+1}}\Delta^2 y_{n+1}\right)\left[\frac{f\left(y_{n-\tau_n}\right) - f\left(y_{n+1-\tau_{n+1}}\right)}{f\left(y_{n-\tau_n}\right) f\left(y_{n+1-\tau_{n+1}}\right)}\right]$$
$$\leq \frac{\Delta^2\left(\frac{1}{p_n}\Delta^2 y_n\right)}{f\left(y_{n-\tau_n}\right)} = -q_n.$$

That is,

$$\Delta v_n \leq -q_n. \tag{2.11}$$

Summing up (2.11) from n_1 to n, we have

$$v_{n+1} - v_{n_1} \leq -\sum_{s=n_1}^{n} q_s.$$

By assumption, we have

$$\lim_{n\to\infty} v_n = -\infty, \tag{2.12}$$

which implies,

$$f\left(y_{n-\tau_n}\right) > 0 \text{ and hence } y_{n-\tau_n} > 0. \tag{2.13}$$

By equation (2.12), there exists a positive integer $n_2 \geq n_1$ such that
$v_n \leq -M+1, M>0, \ n \geq n_2.$

That is,

$$\frac{\Delta\left(\frac{1}{p_n}\Delta^2 y_n\right)}{f\left(y_{n-\tau_n}\right)} = v_n \leq -M+1.$$

This implies

$$\Delta\left(\frac{1}{p_n}\Delta^2 y_n\right) + (M+1) f\left(y_{n-\tau_n}\right) \leq 0, \quad \text{for} \quad n \geq n_2. \tag{2.14}$$

Let

$$\lim_{n\to\infty} y_n = L.$$

Then $L \geq 0$.

Case(i): $L > 0$

If $L > 0$, by the continuity of f, we have

$$\lim_{n\to\infty} f\left(y_{n-\tau_n}\right) = f(L) > 0.$$

So we can choose n_3 such that

$$f\left(y_{n-\tau_n}\right) > \frac{1}{2} f(L), \ n \geq n_3. \tag{2.15}$$

Substituting (2.15) in (2.14), we get

$$\Delta\left(\frac{1}{p_n}\Delta^2 y_n\right) + \frac{1}{2} f(L)(M+1) \leq 0,$$

which implies

$$\Delta\left(\frac{1}{p_n}\Delta^2 y_n\right) \leq -\frac{1}{2} f(L)(M+1).$$

That is,

$$\Delta\left(\frac{1}{p_n}\Delta^2 y_n\right) \leq -k_1, \quad \text{where} \quad k_1 = \frac{1}{2} f(L)(M+1) > 0.$$

Summing the last inequality from n_3 to $n-1$, we have

$$\frac{1}{p_n}\Delta^2 y_n \le \frac{1}{p_{n_3}}\Delta^2 y_{n_3} - k(n-n_3).$$

This implies

$$\frac{1}{p_n}\Delta^2 y_n \to -\infty \quad \text{as} \quad n \to \infty.$$

Proceeding in this way, finally we get

$$y_n \to -\infty \text{ as } n \to \infty.$$

which contradicts (2.13).

Case(ii): $L = 0$

If $L = 0$, then $\lim_{n\to\infty} y_n = 0$.

That is $y_n \to 0$ as $n \to \infty$,

which contradicts (2.13).

The proof of the Theorem is now complete.

3 Examples

Example 3.1 *Consider the difference equation*

$$\Delta^2\left(n\Delta^2 y_n\right) + 16(n+1)y_{n+1} = 0.$$

Here $p_n = \dfrac{1}{n}$, $q_n = 16(n+1)$ and $\sum_{n=0}^{\infty} q_n = \infty$.

All the conditions of Theorem 2.1 are satisfied. Hence all the solutions of the equation (3.1)are oscillatory.

One of the solutions is $y_n = (-1)^n$.

Example 3.2 *Consider the difference equation*

$$\Delta^2\left((n+3)\Delta^2 y_n\right) + \frac{16(16n+72)}{3} y_{n+1} = 0.$$

Here $p_n = \dfrac{1}{n+3}$, $q_n = \dfrac{16(16n+72)}{3}$ and

$$\sum_{k=0}^{\infty} \frac{(k-N)}{R_k} q_k = \sum_{k=0}^{\infty} \frac{16(k-N)(k-3)(16k+72)}{3} = \infty.$$

All the conditions of Theorem 2.2 are satisfied. Hence all the solutions of the equation (3.2)are oscillatory.

One of the solutions is $y_n = (-3)^n$.

Example 3.3 *Consider the difference equation*

$$\Delta^2\left(\frac{1}{n}\Delta^2 y_n\right) + \frac{4\left(4n^3 + 12n^2 + 8n + 1\right)}{n^4 + 4n^3 + 4n^2 + n} y_{n-3} = 0.$$

Here $p_n = n$, $q_n = \frac{4n^3 + 12n^2 + 8n + 1}{n^4 + 4n^3 + 4n^2 + n}$ and $\sum_{n=0}^{\infty} q_n = \infty$.

All the conditions of Theorem 2.3 are satisfied. Hence the difference

$$\Delta\left(\frac{1}{p_n}\Delta^2 y_n\right) = 2(-1)^{n+4}\left[\frac{4n^3 + 12n^2 + 8n + 1}{n^4 + 4n^3 + 4n^2 + n}\right]$$

of the solution $y_n = \frac{(-1)^n}{2}$ of the equation (3.3) oscillates.

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