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Double Commutativity and Some Results on Quasihyponormal Operators

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Abstract

Is well known that even the class of normal operators in Hilbert space is not closed about the sum and product of operators, that is, if A, B are normal operators then $A+B$ and AB are not necessary normal operators. Sum and product will be normal operators if operator A commute with operator B, that is, if $AB = BA$. If we weaken the normality conditions and look at sum and product of new obtained classes of operators (like class of hyponormal or quasihyponormal operators that contains a class of normal operators), then standard commutativity condition is not enough. For this and other reasons we take other commutativity conditions like doubly commutativity which in some cases provide closedness and in other cases give some interesting relations. We are focused on a class of quasihyponormal operators.

Keywords: Hilbert space, bounded linear operators, normal operators, hyponormal operators, products of operators.

1 Introduction

Let us denote by H the infinite separable complex Hilbert space and with $B(H)$ the space of all bounded linear operators defined in Hilbert space H. In the following we will mention some known classes of operators defined in Hilbert space H. Let T be an operator in $B(H)$. The operator T is called normal if it satisfies the following condition $T^*T = TT^*$. The operator T is called hyponormal if $T^*T \geq TT^*$. The later condition is equivalent to the condition $\|Tx\| \geq \|T^*x\|$, for all x in H. We say that an operator T is quasihyponormal if the following condition $T^{*2}T^2 \geq (T^*T)^2$ holds. The foregoing condition is equivalent to $\|T^2x\| \geq \|T^*Tx\|$ for all x in H. Two operators T

and S on H are said to be unitary equivalent if we can find a unitary operator U such that $T = U^*SU$. We say that operators T and S are doubly commutative if $AB = BA$ and $AB^* = B^*A$.

By Fuglede-Putnam theorem we can prove that if normal operators A and B commutes, that is, if $AB = BA$, then $A + B$ and AB are normal operators.

In [3], Proposition 2.3, is shown that if A and B are hyponormal and $AB^* = B^*A$ then products AB and BA are hyponormal operators.

Patel in [5] has proved following

Theorem: Let A be a hyponormal operator and let B be *-paranormal operator. If A and B are doubly commutative (i.e. $AB = BA$ and $AB^* = B^*A$) than the product AB is a *-paranormal operator.

In [2], Problem 9.6, is shown that if A is quasihyponormal, then so is every operator unitarily equivalent to it.

All this above suggest that weakening of normality conditions have much impact, and to obtain some similar results we need more different conditions on commutativity of operators of the same or from different classes.

In this paper we examine some relations about quasihyponormal operators mainly based on results above.

2 Main results

Proposition 2. 1. Let A be a quasihyponormal operator and B let be a hyponormal one. If A double commute with B, then AB and BA are quasihyponormal operators.

Proof. We need to prove that $[(AB)^*(AB)]^2 \le (AB)^{*2}(AB)^2$. For this let evaluate the left-hand site of this relation. We have

$$\begin{aligned}
[(AB)^*(AB)]^2 &= (AB)^*(AB)(AB)^*(AB) \\
&= B^*A^*ABB^*A^*AB \text{(because of double commutativity we have next steps)} \\
&= B^*B\,\underbrace{A^*AA^*A}_{\le A^{*2}A^2}\,B^*B \\
&\le B^*BA^*A^*AAB^*B \text{(because of double commutativity we have next steps)} \\
&= B^*A^*A^*\underbrace{BB^*}_{\le B^*B}AAB \\
&\le B^*A^*A^*B^*BAAB \\
&= B^*A^*B^*A^*BAAB \\
&= B^*A^*B^*A^*ABAB \\
&= (AB)^{*2}(AB)^2 \qquad +
\end{aligned}$$

In similar way we have that BA is quasihyponormal operator.

Proposition 2.2: If A and B are quasihyponormal double commutative operators, then AB and BA are quasihyponormal operators.

Proof:

$$\begin{aligned}
\|(AB)^*(AB)x\| &= \|B^*A^*ABx\| \;(B \text{ commutes with } A \text{ and } A^*)\\
&= \|B^*BA^*Ax\| \;(B \text{ is quasihyponormal operator})\\
&\le \|B^2A^*Ax\| \;(B \text{ commutes with } A \text{ and } A^*)\\
&= \|A^*AB^2x\| \;(A \text{ is quasihyponormal operator})\\
&\le \|A^2B^2x\| \;(A \text{ commutes with } B)\\
&= \|(AB)^2x\|
\end{aligned}$$

In similar way we can prove that BA is a quasihyponormal operator. +

Proposition 2.3 Let A be a quasihyponormal operator such that doubly commutes with an isometric operator B. Then AB is quasihyponormal.

Proof:

$$\begin{aligned}
[(AB)^*(AB)]^2 &= (AB)^*(AB)(AB)^*(AB)\\
&= \underbrace{B^*B}_{I}A^*AA^*A\underbrace{B^*B}_{I}\\
&= \underbrace{A^*AA^*A}_{\le A^{*2}A^2}\\
&\le A^*A^*\underbrace{B^*\underbrace{(B^*B)}_{I}B}_{I}AA\\
&= B^*A^*B^*A^*ABAB\\
&= (AB)^{*2}(AB)^2 \qquad +
\end{aligned}$$

Also, this claim we can prove by norm as following

$$\begin{aligned}
\left\|(AB)^*(AB)x\right\| &= \left\|B^*A^*ABx\right\| (A \text{ doubly commutes with } B)\\
&= \left\|(B^*B)(A^*A)x\right\| (\text{norm properties})\\
&\le \left\|B^*B\right\|\left\|A^*Ax\right\| (B^*B = I \text{ and so } \left\|B^*B\right\| = \left\|I\right\| = 1)\\
&= \left\|A^*Ax\right\| (A \text{ is quasihyponormal operator})\\
&\le \left\|A^2x\right\| (\text{since } B \text{ is isometric operator is true that } \|B^2x\| = \|x\|)\\
&= \|B^2A^2x\| \;(A \text{ doubly commute with } B)\\
&= \|ABABx\|\\
&= \|(AB)^2x\| \qquad +
\end{aligned}$$

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