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IFSM's generated by IFS's

Pearly P. John[1] and Paul Isaac[2]

Department of Mathematics, Bharata Mata College, Thrikkakara
Kochi-682 021, Kerala, India
[1]Corresponding author; e-mail: pearlybmc@gmail.com
[2]e-mail: pibmct@gmail.com

Abstract

The concept of an intuitionistic fuzzy set, which is a generalization of the concept of a fuzzy set, was introduced by Krassimir T. Atanassov in 1986. In this article, we study about the intuitionistic fuzzy submodule (IFSM) generated by an intuitionistic fuzzyset (IFS) in an R -module M and investigate some related properties. Also we discuss the notion of equivalent intuitionistic fuzzy sets in an R-module M .

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1 Introduction

As a generalization of a fuzzy set, the concept of an intuitionistic fuzzy set was introduced by K. T. Atanassov [1,2]. Applying this concept to algebra, B. Davvas-etal. [3] established the intuitionistic fuzzification of the concept of submodules of an R-module. In this paper , in section 2 we give the essential preliminaries and in section 3 we introduce intuitionistic fuzzy submodule generated by an intuitionistic fuzzy set in an R-module M and using this concept we investigate some related properties. In section 4 we study the notion of equivalent intuitionistic fuzzy sets in an R-module M and prove some results.

Throughout this paper, we denote by I the unit interval [0, 1], by R a commutative ring with unity 1 and by M a unitary R- module. $\vee$ denotes the maximum, and $\wedge$ the minimum in the unit interval [0, 1].

2 Preliminaries

In this section we give some basic definitions and results which are used in the sequel. For knowledge regarding modules and fuzzy modules we refer the books by Hungerford [4] and Mordeson & Malik [6] respectively.

2.1. Definition ([1]). An intuitionistic fuzzy set (in short IFS) A in a nonempty set X is an object having the form $A = \{(x, \mu_A(x), \nu_A(x)) \,/\, x \in X\}$ where the functions $\mu_A : X \to I$ and $\nu_A : X \to I$ denote respectively the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) of each element $x \in X$ to the set A, and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for all $x \in X$.

For the sake of simplicity, we will denote the set of all IFS's in X as IFS(X).

2.2. Definition ([1]). Let X be a non-empty set and $A = (\mu_A, \nu_A)$, $B = (\mu_B, \nu_B)$ be IFS's in X. Then

1. $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all $x \in X$
2. $A = B$ if and only if $\mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)$ for all $x \in X$
3. $A^C = (\nu_A, \mu_A)$
4. $A \cap B = \{(x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x)) : x \in X\}$
5. $A \cup B = \{(x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)) : x \in X\}$

2.3. Definition ([7]). A fuzzy set μ in M is called a fuzzy submodule of M if for every $x, y \in M$ and $r \in R$, the following conditions are satisfied

1. $\mu(0) = 1$
2. $\mu(x + y) \geq \mu(x) \wedge \mu(y)$
3. $\mu(rx) \geq \mu(x)$

2.4. Definition ([3]). Let M be a module over a ring R. An IFS $A = (\mu_A, \nu_A)$ in M is called an intuitionistic fuzzy submodule (IFSM) of M if

1. $\mu_A(0) = 1$ and $\nu_A(0) = 0$
2. $\mu_A(x + y) \geq \mu_A(x) \wedge \mu_A(y)\ \forall x, y \in M$
3. $\nu_A(x + y) \leq \nu_A(x) \vee \nu_A(y)\ \forall x, y \in M$
4. $\mu_A(rx) \geq \mu_A(x)\ \forall x \in M,\ \forall r \in R$
5. $\nu_A(rx) \leq \nu_A(x)\ \forall x \in M,\ \forall r \in R$

Remark. By saying that $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy module (IFM) we mean that $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy submodule of some R-module M, denote as $A \in \mathrm{IFSM}(M)$.

2.5. Definition ([5]). Let X be a non-empty set and $A = (\mu_A, \nu_A)$ be an IFS in X, and $\alpha, \beta \in [0, 1]$ be such that $\alpha + \beta \leq 1$. Then the (α, β)-level set of A is defined as $A_{(\alpha,\beta)} = \{x \in X : \mu_A(x) \geq \alpha, \nu_A(x) \leq \beta\}$.

2.6. Definition ([9]). Let X be a non-empty set. The intuitionistic fuzzy point $\hat{1}_{\{0\}}$ in X is defined as $\hat{1}_{\{0\}} = (\mu_{\hat{1}_{\{0\}}}, \nu_{\hat{1}_{\{0\}}})$ where

$$\mu_{\hat{1}_{\{0\}}}(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases} \quad \text{and} \quad \nu_{\hat{1}_{\{0\}}}(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \neq 0 \end{cases} \quad \forall\, x \in X.$$

2.7. Definition ([9]). Let $A_i = (\mu_{A_i}, \nu_{A_i})$ $(i \in J, |J| > 1)$, be a family of IFSM's of *M*. Then $\sum_{i \in J} A_i = \{(x, \mu_{\sum_{i\in J} A_i}(x), \nu_{\sum_{i \in J} A_i}(x)) : x \in M\}$, where,

$$\mu_{\sum_{i\in J} A_i}(x) = \vee \{ \wedge_{i\in J}\, \mu_{A_i}(x_i) : x_i \in M,\ i \in J,\ \textstyle\sum_{i \in J} x_i = x\} \ \ \forall\, x \in M,$$

and $$\nu_{\sum_{i\in J} A_i}(x) = \wedge \{ \vee_{i\in J}\, \nu_{A_i}(x_i) : x_i \in M,\ i \in J,\ \textstyle\sum_{i \in J} x_i = x\} \ \ \forall\, x \in M,$$

where, in $\sum_{i \in J} x_i$, at most finitely many x_i's are not equal to zero. $\sum_{i \in J} A_i$ is called the weak sum of the A_i's.

2.8. Definition ([6]). Let μ be a fuzzy subset in *M*. Then $\cap \{ \nu : \mu \subseteq \nu,\ \nu$ is fuzzy submodule of $M\}$ is a fuzzy submodule of *M*, called the fuzzy submodule generated by the fuzzy subset μ and denoted by $\langle \mu \rangle$.

2.9. Definition ([10]). Two fuzzy subsets μ, ν in an *R*-module *M* are said to be equivalent if $\langle \mu \rangle = \langle \nu \rangle$.

3 IFSM generated by an IFS

In this section we study about the IFSM generated by an IFS in an *R*-module *M* and their properties. Also we prove some related results.

3.1. Theorem ([8]). *Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be IFSM's of M, then $A \cap B = (\mu_{A\cap B}, \nu_{A\cap B})$ is also an IFSM of M.*

3.2. Corollary. *Let $A_i = (\mu_{A_i}, \nu_{A_i})$ $(i \in J, |J| > 1)$, be a family of IFSM's of an R-module M. Then $\bigcap_{i \in J} A_i$ is an IFSM of M.*

3.3. Theorem ([9]). *Let $A_i = (\mu_{A_i}, \nu_{A_i})$ $(i \in J, |J| > 1)$, be a family of IFSM's of an R-module M. Then $\sum_{i \in J} A_i$ is an IFSM of M.*

3.4. Definition. Let $A = (\mu_A, \nu_A)$ be an IFS in an *R*-module *M*. Then $\cap \{B : A \subseteq B$, $B \in$ IFSM $(M)\}$ is an IFSM of *M*, called the intuitionistic fuzzy submodule generated by the IFS *A*, and is denoted by $\langle A \rangle$.

3.5. Definition. Let $B = (\mu_B, \nu_B)$ be an IFSM of *M* such that $B = \langle A \rangle$ for some IFS $A = (\mu_A, \nu_A)$ in *M*. Then *A* is called a generating IFS of *B*.

Result:- Let *A*, *B* be IFS's in *M*. Then the following can be verified.
$A \in$ IFSM $(M) \Leftrightarrow \langle A \rangle = A$
$A \subseteq B \Rightarrow \langle A \rangle \subseteq \langle B \rangle$
$\langle A/N \rangle \subseteq \langle A \rangle / N \in$ IFSM (N) where *N* is a submodule of *M*, A/N and $\langle A \rangle / N$ are

the restrictions of A and $\langle A\rangle$ to N respectively.

3.6. Theorem. *Let* $A_i = (\mu_{A_i}, \nu_{A_i})$ $(i \in J, |J| > 1)$ *, be a family of IFSM's of an R-module M. Then* $\langle \bigcup_{i\in J} A_i\rangle = \sum_{i\in J} A_i$.

Proof. We have $\sum_{i\in J} A_i = (\mu_{\sum_{i\in J} A_i}, \nu_{\sum_{i\in J} A_i})$ is an IFSM of M (by theorem 3.3) , where

$$\mu_{\sum_{i\in J} A_i}(x) = \vee \{ \wedge_{i\in J} \mu_{A_i}(x_i) : x_i \in M,\ i \in J,\ \textstyle\sum_{i\in J} x_i = x\}$$

and $\nu_{\sum_{i\in J} A_i}(x) = \wedge \{ \vee_{i\in J} \nu_{A_i}(x_i) : x_i \in M,\ i \in J,\ \sum_{i\in J} x_i = x\}$ $\forall x \in M$

where, in $\sum_{i\in J} x_i$, at most finitely many x_i 's are not equal to zero.

Therefore $\mu_{A_i}(x) \le \mu_{\sum_{i\in J} A_i}(x)$ and $\nu_{A_i}(x) \ge \nu_{\sum_{i\in J} A_i}(x)$ $\forall x \in M,\ i \in J$

Hence $A_i \subseteq \sum_{i\in J} A_i$ $\forall i \in J$, that is $\sum_{i\in J} A_i$ is an IFSM that contains all A_i 's.

Now to show that it is the smallest IFSM that contains all A_i 's , let $B = (\mu_B, \nu_B)$ be any IFSM of M which contains all A_i 's. That is $A_i \subseteq B$ $\forall\ i \in J$, which means that $\mu_{A_i}(x) \le \mu_B(x)$ and $\nu_{A_i}(x) \ge \nu_B(x)$ $\forall x \in M,\ i \in J$.

Let $x \in M$ where $x = \sum_{i\in J} x_i$, $x_i \in M$, then

$$\begin{aligned}\mu_{\sum_{i\in J} A_i}(x) &= \vee \{ \wedge_{i\in J} \mu_{A_i}(x_i) : x_i \in M,\ i \in J,\ \textstyle\sum_{i\in J} x_i = x\}\\ &\le \vee \{ \wedge_{i\in J} \mu_B(x_i) : x_i \in M,\ i \in J,\ \textstyle\sum_{i\in J} x_i = x\}\\ &\le \vee \{ \mu_B(\textstyle\sum_{i\in J} x_i) : x_i \in M,\ i \in J,\ \textstyle\sum_{i\in J} x_i = x\}\\ &= \mu_B(x).\end{aligned}$$

Similarly we can obtain $\nu_{\sum_{i\in J} A_i}(x) \ge \nu_B(x)$ $\forall x \in M$.

Therefore $\sum_{i\in J} A_i \subseteq B$. Thus $\sum_{i\in J} A_i$ is the smallest IFSM which contains all A_i 's, hence $\sum_{i\in J} A_i$ is the smallest IFSM which contains $\bigcup_{i\in J} A_i$.

Therefore $\langle \bigcup_{i\in J} A_i\rangle = \sum_{i\in J} A_i$. Hence the theorem.

3.7. Definition. Let $C = (\mu_C, \nu_C)$ be an IFS in a ring R and $A = (\mu_A, \nu_A)$ be an IFS in an R-module M. Define $C\,.\,A$ and $C \odot A$ as IFS's in M as follows

1. $C\,.\,A = (\mu_{C.A}, \nu_{C.A})$ where
   $\mu_{C.A}(x) = \vee\{\mu_C(r) \wedge \mu_A(y) : r \in R,\ y \in M,\ ry = x\}$ and
   $\nu_{C.A}(x) = \wedge\{\nu_C(r) \vee \nu_A(y) : r \in R,\ y \in M,\ ry = x\}$ $\forall x \in M$.
2. $C \odot A = (\mu_{C\odot A}, \nu_{C\odot A})$ where

$\mu_{C\odot A}(x) = \vee\{\wedge_{i=1}^{n} (\mu_C(r_i) \wedge \mu_A(x_i)) : r_i \in R,\ x_i \in M, 1 \le i \le n, n \in N,$
$\Sigma_{i=1}^{n} r_i x_i = x\}$ and

$\nu_{C\odot A}(x) = \wedge\{\vee_{i=1}^{n} (\nu_C(r_i) \vee \nu_A(x_i)) : r_i \in R,\ x_i \in M, 1 \le i \le n, n \in N,$
$\Sigma_{i=1}^{n} r_i x_i = x\}$ $\forall x \in M$.

3.8. Theorem. *Let* $A = (\mu_A, \nu_A)$ *be an IFS in M, then*

1. *For all* $r \in R$, $\hat{1}_{\{r\}}\,.\,A = rA$
2. *For all* $r \in R$, $x \in M$, $\hat{1}_{\{r\}} \odot A = (\mu_{\hat{1}_{\{r\}}\odot A}, \nu_{\hat{1}_{\{r\}}\odot A})$ *where*

$\mu_{\hat{1}_{\{r\}}\odot A}(x) = \vee\{\wedge_{i=1}^{n} \mu_A(x_i) : x_i \in M, 1 \le i \le n, n \in N, r\Sigma_{i=1}^{n} x_i = x\}$

and $\quad \nu_{\hat{1}_{\{r\}} \odot A}(x) = \wedge\{\vee_{i=1}^{n} \nu_A(x_i) \ : \ x_i \in M, 1 \le i \le n, n \in N \ , r\Sigma_{i=1}^{n} x_i = x\}$.

Proof. For $r \in R$, the IFP $\hat{1}_{\{r\}} = (\mu_{\hat{1}_{\{r\}}}, \nu_{\hat{1}_{\{r\}}})$ in R is defined by

$$\mu_{\hat{1}_{\{r\}}}(s) = \begin{cases} 1 & \text{if } s = r \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \nu_{\hat{1}_{\{r\}}}(s) = \begin{cases} 0 & \text{if } s = r \\ 1 & \text{otherwise} \end{cases}$$

For any $r \in R$, $\hat{1}_{\{r\}} . A = (\mu_{\hat{1}_{\{r\}} .A}, \nu_{\hat{1}_{\{r\}} .A})$. Now for $x \in M, r \in R$ we have

$$\mu_{\hat{1}_{\{r\}} .A}(x) = \vee\{\mu_{\hat{1}_{\{r\}}}(s) \wedge \mu_A(y) : s \in R, \ y \in M, \ sy = x\}$$

$= \vee\{\mu_A(y) : r \in R, \ y \in M, \ ry = x\}$,by the definition of $\mu_{\hat{1}_{\{r\}}}(s)$

$= \mu_{rA}(x)$

Similarly we get $\nu_{\hat{1}_{\{r\}} .A}(x) = \nu_{rA}(x) \ \forall x \in M$. Hence $\hat{1}_{\{r\}} . A = rA \ \forall r \in R$.

For any $r \in R$, $x \in M$

$$\mu_{\hat{1}_{\{r\}} \odot A}(x) = \vee \{\wedge_{i=1}^{n} (\mu_{\hat{1}_{\{r\}}}(r_i) \wedge \mu_A(x_i)) : r_i \in R, \ x_i \in M, 1 \le i \le n, n \in N, \Sigma_{i=1}^{n} r_i x_i = x\}$$

Clearly , $\quad = \vee\{\wedge_{i=1}^{n} \mu_A(x_i) \ : \ x_i \in M, 1 \le i \le n, n \in N, \ r\Sigma_{i=1}^{n} x_i = x\}$

Similarly we get,

$$\nu_{\hat{1}_{\{r\}} \odot A}(x) = \wedge\{\vee_{i=1}^{n} \nu_A(x_i) \ : \ x_i \in M, 1 \le i \le n, n \in N \ , r\Sigma_{i=1}^{n} x_i = x\}.$$

3.9. Definition. Let $A = (\mu_A, \nu_A)$ be an IFS in a ring R. Then A is called an intuitionistic fuzzy ideal (IFI) of R if it satisfies the conditions

1. $\mu_A(x - y) \ge \mu_A(x) \wedge \mu_A(y)$
   $\nu_A(x - y) \le \nu_A(x) \vee \nu_A(y) \quad \forall x, y \in R$ and
2. $\mu_A(xy) \ge \mu_A(x) \vee \mu_A(y)$
   $\nu_A(xy) \le \nu_A(x) \wedge \nu_A(y) \quad \forall x, y \in R$. We denote the set of all IFI's of R by IFI (R) .

3.10. Theorem. *Let $A = (\mu_A, \nu_A)$ be an IFI of R and $B = (\mu_B, \nu_B)$ be an IFSM of M. Then $A \odot B \in$ IFSM (M) .*

Proof. By the definition we have

$$\mu_{A \odot B}(0) = \vee \{\wedge_{i=1}^{n} (\mu_A(r_i) \wedge \mu_B(x_i)) : r_i \in R, \ x_i \in M, 1 \le i \le n, n \in N, \Sigma_{i=1}^{n} r_i x_i = 0\}$$

$= 1$ when $r_i = x_i = 0 \ \ \forall i = 1$ to n, since $\mu_A(0) \ge \mu_A(r) \ \forall r \in R$

Similarly we get $\nu_{A \odot B}(0) = 0$.

Now for any $r \in R, \ x \in M$,

$$\mu_{A \odot B}(rx) = \vee \{\wedge_{i=1}^{n} (\mu_A(r_i) \wedge \mu_B(x_i)) : r_i \in R, \ x_i \in M, 1 \le i \le n, n \in N, \Sigma_{i=1}^{n} r_i x_i = rx\}$$

$$\ge \vee \{\wedge_{i=1}^{n} (\mu_A(rs_i) \wedge \mu_B(z_i)) : s_i \in R, \ z_i \in M, 1 \le i \le n, n \in N, r\Sigma_{i=1}^{n} s_i z_i = rx\}$$

$$\ge \vee \{\wedge_{i=1}^{n} (\mu_A(s_i) \wedge \mu_B(z_i)) : s_i \in R, \ z_i \in M, 1 \le i \le n, n \in N, \Sigma_{i=1}^{n} s_i z_i = x\}$$

(In particular when $r = 1$ on RHS, also since $A = (\mu_A, \nu_A) \in$ IFI (R), $\mu_A(rs_i) \ge \mu_A(r) \vee \mu_A(s_i) \ge \mu_A(s_i)$)

$= \mu_{A\odot B}(x)$.

Similarly we get $\nu_{A\odot B}(rx) \le \nu_{A\odot B}(x)$ $\forall\, x \in M,\ r \in R$.

Now to show that $\mu_{A\odot B}(x+y) \ge \mu_{A\odot B}(x) \wedge \mu_{A\odot B}(y)$ $\forall\, x, y \in M$, let $x, y \in M$ we have

$$\mu_{A\odot B}(x+y) = \vee\ \{\wedge_{i=1}^{n} (\mu_A(r_i) \wedge \mu_B(z_i)) : r_i \in R,\ z_i \in M, 1 \le i \le n, n \in N, \textstyle\sum_{i=1}^{n} r_i z_i = x+y\}$$
$$\ge \vee\ \{\wedge_{i=1}^{n} (\mu_A(s_i) \wedge \mu_B(x_i+y_i)) : s_i \in R,\ x_i, y_i \in M, 1 \le i \le n, n \in N,\ \textstyle\sum_{i=1}^{n} s_i(x_i+y_i) = x+y\}$$
$$\ge \vee\ \{\wedge_{i=1}^{n} (\mu_A(s_i) \wedge (\mu_B(x_i) \wedge \mu_B(y_i))) : s_i \in R,\ x_i, y_i \in M, 1 \le i \le n, n \in N, \textstyle\sum_{i=1}^{n} s_i(x_i+y_i) = x+y\}$$
$$(\text{ since } B = (\mu_B, \nu_B) \in \text{IFSM}(M)\ )$$

$$= \vee\{\wedge_{i=1}^{n} ((\mu_A(s_i) \wedge \mu_B(x_i)) \wedge (\mu_A(s_i) \wedge \mu_B(y_i))) : s_i \in R, x_i, y_i \in M, 1 \le i \le n, n \in N, \textstyle\sum_{i=1}^{n} s_i(x_i+y_i) = x+y\}$$
$$\ge \vee\{(\wedge_{i=1}^{n} (\mu_A(s_i) \wedge \mu_B(x_i)) : s_i \in R, x_i \in M, 1 \le i \le n, n \in N, \textstyle\sum_{i=1}^{n} s_i x_i = x) \wedge (\wedge_{i=1}^{n} (\mu_A(s_i) \wedge \mu_B(y_i)): s_i \in R, y_i \in M, 1 \le i \le n, n \in N, \textstyle\sum_{i=1}^{n} s_i y_i = y)\}$$
$$= \left(\vee\left\{\wedge_{i=1}^{n} \left(\mu_A(s_i) \wedge \mu_B(x_i)\right): s_i \in R, x_i \in M, 1 \le i \le n, n \in N, \textstyle\sum_{i=1}^{n} s_i x_i = x\right\}\right)$$
$$\wedge\ (\vee\{\wedge_{i=1}^{n} \left(\mu_A(s_i) \wedge \mu_B(y_i)\right): s_i \in R, y_i \in M, 1 \le i \le n, n \in N \textstyle\sum_{i=1}^{n} s_i y_i = y\})$$
$$= \mu_{A\odot B}(x) \wedge \mu_{A\odot B}(y) \quad \forall\, x, y \in M$$

Similarly we can get $\nu_{A\odot B}(x+y) \le \nu_{A\odot B}(x) \vee \nu_{A\odot B}(y)$ $\forall x, y \in M$

Hence $A \odot B$ is an IFSM of *M*.

Note: *Let M = R. From above theorem if A , B ∈ IFI (R), then $A \odot B$ ∈ IFI (R).*

3.11. Theorem. *Let $A = (\mu_A, \nu_A)$ be an IFS in M . Define $B = (\mu_B, \nu_B)$, an IFS in M as follows:* $\forall x \in M$ $\mu_B(x) = 1$ *if* $x = 0$,

$$\mu_B(x) = \vee\ \{\wedge_{i=1}^{n} \mu_A(x_i) : r_i \in R, x_i \in M, 1 \le i \le n, n \in N, \textstyle\sum_{i=1}^{n} r_i x_i = x\} \quad \textit{otherwise.}$$

And $\nu_B(x) = 0$ *if* $x = 0$,

$$\nu_B(x) = \wedge\ \{\vee_{i=1}^{n} \nu_A(x_i) : r_i \in R, x_i \in M, 1 \le i \le n, n \in N, \textstyle\sum_{i=1}^{n} r_i x_i = x\} \quad \textit{otherwise.}$$

That is $B = \hat{1}_{\{0\}} \cup (\hat{1}_R \odot A)$. *Then* $\langle A \rangle = B$.

Proof. Clearly $A \subseteq B$ since $\forall\, x \in M$, $\mu_A(x) \le \mu_B(x)$ and $\nu_A(x) \ge \nu_B(x)$.

By definition $\mu_B(0) = 1$ and $\nu_B(0) = 0$.

Let $r \in R,\ x \in M$.

If $rx = 0$ then $\mu_B(rx) = 1 \ge \mu_B(x)$ and $\nu_B(rx) = 0 \le \nu_B(x)$.

Suppose $rx \ne 0$ then $x \ne 0$ and

$$\mu_B(rx) = \vee\ \{\wedge_{i=1}^{n} \mu_A(x_i) : r_i \in R, x_i \in M, 1 \le i \le n, n \in N, \textstyle\sum_{i=1}^{n} r_i x_i = rx\}$$
$$\ge \vee\ \{\wedge_{i=1}^{n} \mu_A(x_i) : s_i \in R, x_i \in M, 1 \le i \le n, n \in N, \textstyle\sum_{i=1}^{n} r s_i x_i = rx\}$$
$$= \vee\ \{\wedge_{i=1}^{n} \mu_A(x_i) : s_i \in R, x_i \in M, 1 \le i \le n, n \in N, r(\textstyle\sum_{i=1}^{n} s_i x_i) = rx\}$$
$$\ge \vee\ \{\wedge_{i=1}^{n} \mu_A(x_i) : s_i \in R, x_i \in M, 1 \le i \le n, n \in N,\ \textstyle\sum_{i=1}^{n} s_i x_i = x\}$$

(In particular when $r = 1$)

$= \mu_B(x)$.

Similarly we can get $\nu_B(rx) \leq \nu_B(x) \;\; \forall \; r \in R, \; x \in M$.

Now let $x, y \in M$. If x, y or $x + y$ equal to zero then clearly $\mu_B(x + y) \geq \mu_B(x) \wedge \mu_B(y)$ and $\nu_B(x + y) \leq \nu_B(x) \vee \nu_B(y)$.

Suppose x, y and $x + y$ are all not equal to zero, then

$$\begin{aligned}
\mu_B(x + y) &= \vee \{\wedge_{i=1}^{n} \mu_A(z_i) : r_i \in R, z_i \in M, 1 \leq i \leq n, n \in N, \textstyle\sum_{i=1}^{n} r_i z_i = x + y\} \\
&\geq \vee \{ \wedge_{i=1}^{n} \mu_A(z_i) : z_i = x_i + y_i,\; x_i, y_i \in M, s_i \in R, 1 \leq i \leq n, n \in N, \\
&\qquad\qquad \textstyle\sum_{i=1}^{n} s_i(x_i + y_i) = x + y\} \\
&= \vee \{(\wedge_{i=1}^{n} \mu_A(x_i)) \wedge (\wedge_{i=1}^{n} \mu_A(y_i)) : s_i \in R,\; x_i, y_i \in M, 1 \leq i \leq n, \\
&\qquad\qquad n \in N, \textstyle\sum_{i=1}^{n} s_i x_i + \sum_{i=1}^{n} s_i y_i = x + y\} \\
&\geq \vee \{(\wedge_{i=1}^{n} \mu_A(x_i) : s_i \in R, x_i \in M, 1 \leq i \leq n, n \in N, \textstyle\sum_{i=1}^{n} s_i x_i = x) \wedge \\
&\qquad (\wedge_{i=1}^{n} \mu_A(y_i) : s_i \in R, y_i \in M, 1 \leq i \leq n, n \in N, \textstyle\sum_{i=1}^{n} s_i y_i = y)\} \\
&= (\vee \{ \wedge_{i=1}^{n} \mu_A(x_i) : s_i \in R, x_i \in M, 1 \leq i \leq n, n \in N, \textstyle\sum_{i=1}^{n} s_i x_i = x\}) \wedge \\
&\qquad (\vee \{ \wedge_{i=1}^{n} \mu_A(y_i) : s_i \in R, y_i \in M, 1 \leq i \leq n, n \in N, \textstyle\sum_{i=1}^{n} s_i y_i = y\}) \\
&= \mu_B(x) \wedge \mu_B(y)
\end{aligned}$$

Similarly we can obtain $\nu_B(x + y) \leq \nu_B(x) \vee \nu_B(y)$ in this case.

Hence $B = (\mu_B, \nu_B)$ is an IFSM of M which contains A. Now let $C = (\mu_C, \nu_C)$ be any IFSM of M which contains A. We will show that $B \subseteq C$. Since $A \subseteq C$, we have $\mu_A(x) \leq \mu_C(x)$ and $\nu_A(x) \geq \nu_C(x) \;\; \forall \; x \in M$. Now

$$\begin{aligned}
\mu_B(x) &= \vee \{\wedge_{i=1}^{n} \mu_A(x_i) : r_i \in R, x_i \in M, 1 \leq i \leq n, n \in N, \textstyle\sum_{i=1}^{n} r_i x_i = x\} \\
&\leq \vee \{\wedge_{i=1}^{n} \mu_C(x_i) : r_i \in R, x_i \in M, 1 \leq i \leq n, n \in N, \textstyle\sum_{i=1}^{n} r_i x_i = x\} \\
&= \mu_C(x). \quad (\text{since } \mu_C(x) = \mu_C\big(\textstyle\sum_{i=1}^{n} r_i x_i\big) \geq \wedge_{i=1}^{n} \mu_C(r_i x_i) \geq \wedge_{i=1}^{n} \mu_C(x_i) \\
&\qquad\qquad \therefore \mu_C(x) = \vee \{\wedge_{i=1}^{n} \mu_C(x_i),\; n \in N\})
\end{aligned}$$

Similarly we can obtain $\nu_B(x) \geq \nu_C(x) \;\; \forall \; x \in M$. Therefore $B \subseteq C$.

Thus we proved $\langle A \rangle = B$.

4 Equivalent IFS's in an *R*-module *M*

In this section we introduce Equivalent IFS's in an R-module M and give a sufficient condition for two IFS's in an R-module M to be equivalent.

4.1. Definition. Two IFS's $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ in an R-module M are said to be equivalent if $\langle A \rangle = \langle B \rangle$.

4.2. Theorem. *Let $A = (\mu_A, \nu_A)$, $B = (\mu_B, \nu_B)$ be IFS's in an R-module M and forevery $\alpha, \beta \in (0, 1]$ with $\alpha + \beta \leq 1$, if $A_{(\alpha,\beta)}$ and $B_{(\alpha,\beta)}$ are equivalent subsets in M, then A and B are equivalent.*

Proof. Let $A_{(\alpha,\beta)}$ and $B_{(\alpha,\beta)}$ be equivalent subsets of M for every $\alpha, \beta \in (0, 1]$ with $\alpha + \beta \leq 1$. Then $\langle A_{(\alpha,\beta)} \rangle = \langle B_{(\alpha,\beta)} \rangle$ where $A_{(\alpha,\beta)} = \{ x \in M : \mu_A(x) \geq \alpha, \nu_A(x) \leq \beta\}$ and $B_{(\alpha,\beta)} = \{ x \in M : \mu_B(x) \geq \alpha, \nu_B(x) \leq \beta\}$ are subsets of M.

We have $\langle A_{(\alpha,\beta)}\rangle = \cap\{C : A_{(\alpha,\beta)} \subseteq C$, C is submodule of $M\}$ and $\langle B_{(\alpha,\beta)}\rangle = \cap\{D : B_{(\alpha,\beta)} \subseteq D$, D is submodule of $M\}$.
Now to show that $\langle A\rangle = \langle B\rangle$, we have to show that $\mu_{\langle A\rangle}(x) = \mu_{\langle B\rangle}(x)$ and $\nu_{\langle A\rangle}(x) = \nu_{\langle B\rangle}(x)\ \forall\ x \in M$.
We have from above theorem $\langle A\rangle = (\mu_{\langle A\rangle}, \nu_{\langle A\rangle})$ where $\mu_{\langle A\rangle}(x) = 1$ and $\nu_{\langle A\rangle}(x) = 0$ when $x = 0$ and When $x \neq 0$,
$\mu_{\langle A\rangle}(x) = \vee\{\wedge_{i=1}^{n} \mu_A(x_i) : r_i \in R, x_i \in M, 1 \le i \le n, n \in N, \Sigma_{i=1}^{n} r_i x_i = x\}$ and $\nu_{\langle A\rangle}(x) = \wedge\{\vee_{i=1}^{n} \nu_A(x_i) : r_i \in R, x_i \in M, 1 \le i \le n, n \in N, \Sigma_{i=1}^{n} r_i x_i = x\}$.
Similarly $\langle B\rangle =$
$(\mu_{\langle B\rangle}, \nu_{\langle B\rangle})$ where $\mu_{\langle B\rangle}(x) = 1$ and $\nu_{\langle B\rangle}(x) = 0$ when $x = 0$ and when $x \neq 0$, $\mu_{\langle B\rangle}(x) = \vee\{\wedge_{j=1}^{m} \mu_B(y_j): s_j \in R, y_j \in M, 1 \le j \le m, m \in N, \sum_{j=1}^{m} s_j y_j = x\}$ and $\nu_{\langle B\rangle}(x) = \wedge\{\vee_{j=1}^{m} \nu_B(y_j) : s_j \in R, y_j \in M, 1 \le j \le m, m \in N, \Sigma_{j=1}^{m} s_j y_j = x\}$.
When $x = 0$ clearly $\langle A\rangle = \langle B\rangle$. When $x \neq 0$, take $r_i \in R, x_i \in M, 1 \le i \le n, n \in N$, $\Sigma_{i=1}^{n} r_i x_i = x$. We let $\alpha = \wedge_{i=1}^{n} \mu_A(x_i)$ and $\beta = \vee_{i=1}^{n} \nu_A(x_i)$, then clearly $\alpha + \beta \le 1$.
So that we get $\mu_A(x_i) \ge \alpha$ and $\nu_A(x_i) \le \beta\ \ \forall\ i = 1$ to n, $n \in N$.
That is $x_i \in A_{(\alpha,\beta)} \forall\ i = 1$ to n, $n \in N$, which implies $x_i \in \langle A_{(\alpha,\beta)}\rangle\ \forall\ i = 1$ to n, $n \in N$.
Therefore $\Sigma_{i=1}^{n} r_i x_i \in \langle A_{(\alpha,\beta)}\rangle$ for $n \in N$, since $\langle A_{(\alpha,\beta)}\rangle$ is a submodule of M. Hence $x \in \langle A_{(\alpha,\beta)}\rangle$. Now since $\langle A_{(\alpha,\beta)}\rangle = \langle B_{(\alpha,\beta)}\rangle$, we get $x \in \langle B_{(\alpha,\beta)}\rangle$, so that there exists $s_j \in R, y_j \in B_{(\alpha,\beta)}, 1 \le j \le m, m \in N$ such that $x = \Sigma_{j=1}^{m} s_j y_j$.
Since $y_j \in B_{(\alpha,\beta)}$, we get $\mu_B(y_j) \ge \alpha$ and $\nu_B(y_j) \le \beta\ \ \forall\ j = 1$ to m, $m \in N$.
That is $\mu_B(y_j) \ge \wedge_{i=1}^{n} \mu_A(x_i)$ and $\nu_B(y_j) \le \vee_{i=1}^{n} \nu_A(x_i)\ \forall\ j = 1$ to m, $m \in N$.
This implies that $\wedge_{j=1}^{m} \mu_B(y_j) \ge \wedge_{i=1}^{n} \mu_A(x_i)$ -------(1) and $\vee_{j=1}^{m} \nu_B(y_j) \le \vee_{i=1}^{n} \nu_A(x_i)$ -------(2) where $\Sigma_{i=1}^{n} r_i x_i = \Sigma_{j=1}^{m} s_j y_j = x$, $m \in N$, $n \in N$.
From (1) we get,

$$\vee\{\wedge_{j=1}^{m} \mu_B(y_j): s_j \in R, y_j \in M, 1 \le j \le m, m \in N, \Sigma_{j=1}^{m} s_j y_j = x\} \ge$$
$$\vee\{\wedge_{i=1}^{n} \mu_A(x_i) : r_i \in R, x_i \in M, 1 \le i \le n, n \in N, \Sigma_{i=1}^{n} r_i x_i = x\}$$

which implies $\mu_{\langle B\rangle}(x) \ge \mu_{\langle A\rangle}(x)\ \forall\ x \in M$. Similarly from (2), we get $\nu_{\langle B\rangle}(x) \le \nu_{\langle A\rangle}(x)\ \forall\ x \in M$. Therefore $\langle A\rangle \subseteq \langle B\rangle$. In similar fashion we can show that
$\langle B\rangle \subseteq \langle A\rangle$. Hence $\langle A\rangle = \langle B\rangle$.

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