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Asymptotic Formulas for Two Restricted Partition Functions Derived from Hardy-Ramanujan-Rademacher Formula

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Abstract

We have defined $e(n)$ to be the number of partitions of n with last two or more parts equal and $u(n)$ is the number of partitions of n with last two parts unequal. Then we have established the identity $p(n) = 2u(n-1) + e(n-1)$. Finally with the help of this identity we have derived asymptotic formulas for both $e(n)$ and $u(n)$.

Keywords. Asymptotic formula, Dedekind sum, Exponential function, Generating function, Hyperbolic function, Mutually exclusive and exhaustive, Partition function, Restricted partition.

AMS subject classification: 05 A 17; 11 P 81

1. Introduction.

The theory of partitions was founded by Leonhard Euler (1707-1783) about two centuries ago. It is an area of additive number theory, a subject concerning the representation of non-negative integers as sums of other positive integers.

Definition 1.1 A partition of a non-negative integer n is a representation of n as a sum of positive integers, called summands or parts of the partition. The order of the summands is irrelevant.

Definition 1.2 The partition function $p(n)$ is defined by the number of partitions of n.

Definition 1.3 Let $e(n)$ be the number of partitions of n with last two or more parts equal.

Definition 1.4 Let $u(n)$ be the number of partitions of n with last two parts unequal.

For example, the partitions of 6 are 6, 5+1, 4+2, 4+1+1, 3+3, 3+2+1, 3+1+1+1, 2+2+2, 2+2+1+1, 2+1+1+1+1, and 1+1+1+1+1+1.

Thus, $p(6) = 11$, $e(6) = 7$, and $u(6) = 4$.

Remark 1.5 Though the order of the summands is irrelevant when writing the partitions of n, for consistency, partitions of n will be written with the summands in a non-increasing order.

Remark 1.6 By convention $p(0) = 1$, and $p(n) = 0$ for negative n.

Euler extended the technique of generating function in 1748 in his *"Introductio in Analysin Infifitorum"* where he uses them to attack the problem of partitions.

The generating function for $p(n)$ is given by

$$\sum_{n=1}^{\infty} p(n)x^n = \prod_{k=1}^{\infty}(1 - x^k).$$

Leonhard Euler's pentagonal number theorem implies

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + \cdots$$

Where the numbers 1, 2, 5, 7... that appear on the right side of the equation are the generalized pentagonal numbers $g_k = \frac{k(3k-1)}{2}$ for nonzero integers k. More formally

$$p(n) = \sum_{k}(-1)^{k-1} p\left(n - \frac{k(3k-1)}{2}\right)$$

Where the summation is over all nonzero integers k (positive and negative) and $p(n)$ is taken to be zero if $n < 0$.

Euler's formula involves recursion and it is efficient for computing $p(n)$ for all n up to some limit. Euler's formula is impractical for evaluating $p(n)$ for an isolated, large n. One of the most astonishing number–theoretical discoveries of the 20th century is the Hardy–Ramanujan–Rademacher formula. It was first given by Hardy and Ramanujan as an asymptotic expression in 1917. They obtained the expansion by an ingenious and intricate calculation involving the singularities of the generating function of $p(n)$ in the unit circle. This formula yields a value which differ from $p(n)$ by a quantity no more than the reciprocal of the square root of n, when calculated up to a certain number of terms. Since $p(n)$ is an integer, the exact value of $p(n)$ is the nearest integer value what is given by the series. For a proof see [4].

Theorem 1.7 [Hardy-Ramanujan] The asymptotic expansions for $p(n)$ is given by

$$p(n) = \frac{1}{2\sqrt{2}}\sum_{k=1}^{v}\sqrt{k}A_k(n)\frac{d}{dn}e^{\left(\pi\sqrt{\frac{2}{3}}\frac{\sqrt{n-\frac{1}{24}}}{k}\right)}$$

Where

$$A_k(n) = \sum_{0 \le m < k; (m,k)=1} e^{\pi i [s(m,k) - \frac{1}{k} 2nm]}$$

And $S(m,k)$ is the Dedekind sum

$$S(m,k) = \sum_{r \bmod k} \left(\left(\frac{r}{k}\right)\right)\left(\left(\frac{mr}{k}\right)\right)$$

$$((x)) = \begin{cases} x - \lfloor x \rfloor - \frac{1}{2}, if\ x \epsilon \mathbb{R} - \mathbb{Z} \\ 0, if\ x \epsilon \mathbb{Z} \end{cases}$$

The series representation of Hardy–Ramanujan is an asymptotic series in the sense when summed up to infinity, it diverges. Subsequently Hans Rademacher noticed that by making a very mild but important change, namely, by replacing the exponential function with hyperbolic functions, the Hardy–Ramanujan asymptotic series could be converted into a series that in fact converges to $p(n)$. The proof of Rademacher's formula involves Ford circles, Farey sequences, modular symmetry and Dedekind eta function in a central way. For a proof see [6].

Theorem 1.8 [Hardy-Ramanujan-Rademacher] The convergent series expansion for $p(n)$ is given by

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n) \frac{d}{dn}\left(\frac{1}{\sqrt{n - \frac{1}{24}}} \sinh\left[\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n - \frac{1}{24}\right)}\right]\right).$$

Hardy-Ramanujan-Rademacher convergent series expansion has as its first term Hardy and Ramanujan asymptotic.

Corollary 1.9

$$p(n) \approx \frac{1}{4n\sqrt{3}} e^{\left(\pi\sqrt{\frac{2n}{3}}\right)}.$$

In our present work we have established the identity $p(n) = 2u(n-1) + e(n-1)$, and with the help of this partition identity we have derived approximate formulas for $e(n)$ and $u(n)$ from the above corollary of Hardy-Ramanujan-Rademacher convergent series expansion.

2. Motivation.

Our motivation is to derive asymptotic formulas for $e(n)$ and $u(n)$ from Hardy-Ramanujan-Rademacher approximation for $p(n)$. We will divide all the partitions of n in to two mutually exclusive and exhaustive classes $e(n)$ and $u(n)$. Then we will establish the partition identity $p(n) = 2u(n-1) + e(n-1)$. Using this identity we will derive asymptotic formulas for $e(n)$ and $u(n)$ from Hardy-Ramanujan-Rademacher approximation.

3. The Main Result.

Our goal is to establish the result $p(n) = 2u(n-1) + e(n-1)$, which is the central part of the present work, and after that we will derive asymptotic formulas for $e(n)$ and $u(n)$. Before that we will prove an important lemma:

Lemma 3.1

$$p(n) = e(n) + u(n).$$

Proof

Since the two classes $e(n)$ and $u(n)$ are mutually exclusive and together they constitute all the partitions of n, the result follows at once. ■

Next result is the most important and crucial part of our present work. In the proof we shall use some combinatorial technique.

Theorem 3.2

$$p(n) = 2u(n-1) + e(n-1).$$

Proof

Let us first write down the partitions of $(n-1)$ as follows:

(n-1)
(n-2)+1
(n-3)+2 (n-3)+1+1
(n-4)+3 (n-4)+2+1 (n-4)+1+1+1
(n-5)+4 (n-5)+3+1 (n-5)+2+2 (n-5)+2+1+1 (n-5)+1+1+1+1
(n-6)+5 (n-6)+4+1 (n-6)+3+2 (n-6)+3+1+1 (n-6)+2+2+1 (n-6)+2+1+1+1 (n-6)+1+1+1+1+1
....

$$\underbrace{1 + 1 + \cdots + 1}_{(n-1) times}.$$

Here we are maintaining some order while listing the partitions of $(n-1)$. That is, the partition $p_1 + p_2 + \cdots + p_i + \cdots + p_l$ will appear before $q_1 + q_2 + \cdots + q_i + \cdots + q_m \ iff \ p_j = q_j$ for $j = 0,1, \dots, i$ and $p_{i+1} > q_{i+1}$ where $0 \le i < l$.

Now if we add 1 to any one of the above partitions of $(n-1)$ we shall get a partition of n.

We can add 1 to a partition of $(n-1)$ in two different ways to get two different partitions of n. First we can add one to the last part of the partition and second we can add 1 as a different part. That is, if $p_1 + p_2 + \cdots + p_i + p_{i+1} + \cdots + p_{l-1} + p_l$ is a partition of $(n-1)$, by adding 1 in two different ways we shall get $p_1 + p_2 + \cdots + p_i + p_{i+1} + \cdots + p_{l-1} + (p_l + 1)$ and $p_1 + p_2 + \cdots + p_i + p_{i+1} + \cdots + p_{l-1} + p_l + 1$

as two different partitions of n. If $p_i > p_{i+1} = p_{i+2} = \cdots = p_{l-1} = p_l$ for some $i(1 \leq i < l)$, then $p_1 + p_2 + \cdots + p_i + p_{i+1} + \cdots + p_{l-1} + (p_l + 1)$ will become $p_1 + p_2 + \cdots + p_i + (p_l + 1) + p_{i+1} + \cdots + p_{l-1}$ as we write the summands of a partition in non-increasing order.

Now we shall consider an arbitrary partition of $(n-1)$, say, $p_1 + p_2 + \cdots + p_i + p_{i+1} + \cdots + p_l + p_m + p_q$. There are two cases, either $p_m > p_q$ or $p_m = p_q$.

Case I. If $p_m > p_q$, then by adding 1 in two different ways we will get $p_1 + p_2 + \cdots + p_i + p_{i+1} + \cdots + p_l + p_m + (p_q + 1)$ and $p_1 + p_2 + \cdots + p_i + p_{i+1} + \cdots + p_l + p_m + p_q + 1$ as two different partitions of n. It is impossible to get these two partitions from any of the preceding partitions of $p_1 + p_2 + \cdots + p_i + p_{i+1} + \cdots + p_l + p_m + p_q$. By doing the reverse thing, that is, by subtracting 1 from the last parts of the above two partitions of n we get $p_1 + p_2 + \cdots + p_i + p_{i+1} + \cdots + p_l + p_m + p_q$ and this proves our claim.

Case II. If $p_m = p_q$, then we will get only one new partition of n, namely $p_1 + p_2 + \cdots + p_i + p_{i+1} + \cdots + p_l + p_m + p_q + 1$. It is impossible to get this partition from any of the preceding partitions of $p_1 + p_2 + \cdots + p_i + p_{i+1} + \cdots + p_l + p_m + p_q$ as by subtracting 1 from the last part of $p_1 + p_2 + \cdots + p_i + p_{i+1} + \cdots + p_l + p_m + p_q + 1$ we get $p_1 + p_2 + \cdots + p_i + p_{i+1} + \cdots + p_l + p_m + p_q$.

Now if $p_i > p_{i+1} = \cdots = p_l = p_m = p_q$, then by adding 1 to the last part we shall get $p_1 + p_2 + \cdots + p_i + p_{i+1} + \cdots + p_l + p_m + (p_q + 1)$ and as we write the summands in a non-increasing order, it will become $p_1 + p_2 + \cdots + p_i + \left(p_q + 1\right) + p_{i+1} + \cdots + p_l + p_m$. Now subtracting 1 from the last part of the partition we get $p_1 + p_2 + \cdots + p_i + \left(p_q + 1\right) + p_{i+1} + \cdots + p_l + (p_m - 1)$. Thus $p_1 + p_2 + \cdots + p_i + \left(p_q + 1\right) + p_{i+1} + \cdots + p_l + p_m$ has already appeared in the list of partitions of n from $p_1 + p_2 + \cdots + p_i + \left(p_q + 1\right) + p_{i+1} + \cdots + p_l + (p_m - 1)$ which is the preceding partition of $p_1 + p_2 + \cdots + p_i + p_{i+1} + \cdots + p_l + p_m + p_q$.

Thus we have seen that from every partition of $(n-1)$ in the class $u(n-1)$ we will get two new partitions of n and from every partition of $(n-1)$ in the class $e(n-1)$ we will get only one new partition of n. Again if we list all the partitions of n and subtract 1 from the last parts of each of the partitions we will get all the partitions of n.

Therefore we can conclude that each partition in the class $e(n-1)$ will give only one new partition of n and each partition in the class $u(n-1)$ will give two new partitions of n and together these constitute all the partitions of n. Thus

$$p(n) = e(n-1) + 2u(n-1).$$

■

From the above theorem we immediately get the following two corollaries:

Corollary 3.3

$$e(n) = 2p(n) - p(n+1).$$

Proof

From lemma 3.1 and theorem 3.2 we get

$$p(n) = 2[u(n-1) + e(n-1)] - e(n-1) = 2p(n-1) - e(n-1).$$

Replacing n by $n+1$ we get the desired result. ■

Corollary 3.4

$$u(n) = p(n+1) - p(n).$$

Proof

From lemma 3.1 and theorem 3.2 we get

$$p(n) = u(n-1) + u(n-1) + p(n-1) = u(n-1) + p(n-1).$$

Replacing n by $n+1$ we get the desired result. ■

Theorem 3.5

$$e(n) \approx \frac{1}{2\sqrt{3}}\left(\frac{1}{n}e^{\pi\sqrt{\frac{2n}{3}}} - \frac{1}{2(n+1)}e^{\pi\sqrt{\frac{2(n+1)}{3}}}\right).$$

Proof

By corollary 1.9 we get

$$p(n+1) \approx \frac{1}{4(n+1)\sqrt{3}}e^{\pi\sqrt{\frac{2(n+1)}{3}}}.$$

$$2p(n) \approx \frac{1}{2n\sqrt{3}}e^{\pi\sqrt{\frac{2n}{3}}}.$$

Now by corollary 3.3 we get

$$e(n) = 2p(n) - p(n+1)$$

$$e(n) \approx \frac{1}{2\sqrt{3}}\left(\frac{1}{n}e^{\pi\sqrt{\frac{2n}{3}}} - \frac{1}{2(n+1)}e^{\sqrt{\frac{2(n+1)}{3}}}\right).$$

This is the desired result. ■

Theorem 3.6

$$u(n) \approx \frac{1}{4\sqrt{3}}\left(\frac{1}{n+1}e^{\pi\sqrt{\frac{2(n+1)}{3}}} - \frac{1}{n}e^{\pi\sqrt{\frac{2n}{3}}}\right).$$

Proof

By corollary 3.4 we get

$$u(n) = p(n+1) - p(n).$$
$$u(n) \approx \frac{1}{4\sqrt{3}}\left(\frac{1}{n+1}e^{\pi\sqrt{\frac{2(n+1)}{3}}} - \frac{1}{n}e^{\pi\sqrt{\frac{2n}{3}}}\right).$$

This is the required result. ∎

4. Conclusion

In this article we have mentioned a technique of listing all the partitions of n from the list of partitions of $(n-1)$ by adding 1 to each of the partitions of $(n-1)$ in two different ways. We can determine $p(n+1)$ from the list of partitions of n by simply calculating any one of $e(n)$ or $u(n)$. But we cannot determine $p(n+2)$ from the list of partitions of n, since we cannot calculate $e(n+1)$ or $u(n+1)$ directly from the list of partitions of n. The asymptotic formulas give approximate values of $e(n)$ and $u(n)$ for large n.

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