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# On Impulsive Nonlocal Integro-Differential Equations with Finite Delay

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**Abstract**

In the present paper, we investigate the existence of mild solutions of an impulsive integro-differential equations with nonlocal condition in Banach spaces. Our analysis is based on semigroup theory and Krasnoselskii-Schaefer type fixed point theorem.

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## 1. Introduction

Impulsive equations arise in many different real processes and phenomena which appears in physics, population dynamics, medicine, economics etc. The study of impulsive functional differential equations is linked to their utility in stimulating processes and phenomena subject to short time perturbations during their evolution. The perturbations are performed discretely and their duration is negligible in comparison with the total duration of the processes and phenomena. That is the reason for the perturbations are considered to take place instantaneously in the form of impulses. Impulsive differential equations in recent years have been an object of investigation with increasing interest. For more information see the monographs, Lakshmikantham [10], Samoilenko and perestyuk [15], the research papers [1], [16] and the references cited therein.

On the other hand, nonlocal condition has better effect on the solution and is more precise for physical measurements than the classical condition. That is the reason why initial value problems with nonlocal conditions have received much attention in recent years, for more information see [1], [4]-[6], [8], [9], [11], [13], [16] and the references cited therein.

In the present paper we consider semilinear functional impulsive integro-differential equation of first order of the type:

$$x'(t) = Ax(t) + f(t, x_t, \int_0^t k(t,s)h(s,x_s)ds), \quad t \in (0,T], \quad t \neq \tau_k, k = 1,2,\ldots,m \tag{1}$$

$$x(t) + (g(x_{t_1},\ldots,x_{t_p}))(t) = \phi(t), \quad -r \leq t \leq 0, \tag{2}$$

$$\Delta x(\tau_k) = I_k x(\tau_k), \quad k = 1,2,\ldots,m, \tag{3}$$

where $0 < t_1 < t_2 < \cdots < t_p \leq T$, $p \in \mathbb{N}$, $A$ is the infinitesimal generator of strongly continuous semigroup of bounded linear operators $\{T(t)\}_{t\geq 0}$ and $I_k (k = 1,2,\ldots,m)$ are the linear operators acting in a Banach space $X$. The functions $f, h, g, k$ and $\phi$ are given functions satisfying some assumptions. The impulsive moments $\tau_k$ are such that $0 \leq \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_m < \tau_{m+1} \leq T$, $m \in \mathbb{N}$, $\Delta x(\tau_k) = x(\tau_k + 0) - x(\tau_k - 0)$, where $x(\tau_k + 0)$ and $x(\tau_k - 0)$ are, respectively, the right and the left limits of $x$ at $\tau_k$.

Equations of the form (1)-(3) or their special forms arise in some physical applications as a natural generalization of the classical initial value problems for eg. see [1], [2], [7], [12], [14]. The results for semilinear functional differential nonlocal problem are extended for the case of impulsive effect.

As usual, in the theory of impulsive differential equations at the points of discontinuity $\tau_i$ of the solution $t \to x(t)$, we assume that $x(\tau_i) \equiv x(\tau_i - 0)$. It is clear that, in general the derivatives $x'(\tau_i)$ do not exist. On the other hand, according to (1) there exist the limits $x'(\tau_i \pm 0)$. According to the above convention, we assume $x'(\tau_i) = x'(\tau_i - 0)$.

The aim of the present paper is to study the existence of mild solution of nonlocal initial value problem for an impulsive functional integro-differential equation. We are generalizing the results reported in [1], [8], [9] for the case of impulse effect. We use semigroup theory and Krasnoselskii-Schaefer type fixed point theorem to obtain our result.

This paper is organized as follows. Section 2 presents the preliminaries and hypotheses. In Section 3, we prove existence of mild solutions and section 4, we give application based on our result.

## 2. Preliminaries and Hypotheses

Let $X$ be a Banach space with the norm $\|\cdot\|$. Let $C = \mathcal{C}([-r,0],X)$, $0 < r < \infty$, be the Banach space of all continuous functions $\psi : [-r,0] \to X$ endowed with supremum norm $\|\psi\|_C = \sup\{\|\psi(t)\| : -r \leq t \leq 0\}$ and $B$ denote the set $\{x : [-r, T] \to X | x(t)$ is continuous at $t \neq \tau_k$,left continuous at $t = \tau_k$, and the right limit $x(\tau_k + 0)$ exists for $k = 1, 2, \ldots, m\}$. Clearly, B is a Banach space with the supremum norm $\|x\|_B = \sup\{\|x(t)\| : t \in [-r, T] \setminus \{\tau_1, \tau_2, \ldots, \tau_m\}\}$. For any $x \in B$ and $t \in [0, T] \setminus \{\tau_1, \tau_2, \ldots, \tau_m\}$, we denote $x_t$ the element of $C$ given by

$$x_t(\theta) = x(t + \theta), \quad \text{for} \quad \theta \in [-r, 0]$$

and $\phi$ is a given element of $C$.

In this paper, we assume that, there exist positive constant $K \geq 1$ such that $\|T(t)\| \leq K$, for every $t \in [0, T]$. Also $k : [0, T] \times [0, T] \to \mathbb{R}$ is continuous function and as the set $[0, T] \times [0, T]$, is compact, there exists a constant $L > 0$ such that $|k(t, s)| \leq L$, for $0 \leq s \leq t \leq T$.

**Definition 2.1.** A function $x \in B$ satisfying the equations:

$$\begin{aligned} x(t) &= T(t)\phi(0) - T(t)(g(x_{t_1}, \ldots, x_{t_p}))(0) \\ &\quad + \int_0^t T(t-s) f(s, x_s, \int_0^s k(s, \tau) h(\tau, x_\tau) d\tau) ds \\ &\quad + \sum_{0 < \tau_k < t} T(t - \tau_k) I_k x(\tau_k), \quad t \in (0, T], \\ x(t) &+ (g(x_{t_1}, \ldots, x_{t_p}))(t) = \phi(t), \quad -r \leq t \leq 0, \end{aligned}$$

is said to be the mild solution of the initial value problem (1)-(3).

**Remark 2.2.** A mild solution of equations (1)-(3) satisfies (2) and (3). However, a mild solution may not be differentiable at zero.

The following inequality will be useful while proving our result.

**Lemma 2.3. ([15, p.12]** Let a nonnegative piecewise continuous function $u(t)$ satisfy for $t \geq t_0$, the inequality

$$u(t) \leq C + \int_{t_0}^t v(s) u(s) ds + \sum_{t_0 < \tau_i < t} \beta_i u(\tau_i)$$

where $C \geq 0$, $\beta_i \geq 0$, $v(t) > 0$, $\tau_i$ are the first kind discontinuity points of the function $u(t)$. Then the following estimate holds for the function $u(t)$,

$$u(t) \leq C \prod_{t_0 < \tau_i < t} (1 + \beta_i) exp(\int_{t_o}^t v(s) ds).$$

The following theorem is known as Krasnoselskii-Schaefer type fixed point theorem.

**Theorem 2.4. [3]** Let $X$ be a Banach space and let $A, B : X \to X$ be two operators satisfying:

i) $A$ is a contraction, and

ii) $B$ is completely continuous continuous.

Then, either,

a) the operator equation $Ax + Bx = x$ has a solution, or

b) the set $\Omega = \{u \in X : \lambda A\left(\frac{u}{\lambda}\right) + \lambda Bu = u, 0 < \lambda < 1\}$ is unbounded.

We list the following hypotheses for our convenience.

$(H_1)$ Let $f : [0,T] \times C \times X \to X$ such that for every $w \in B$, $x \in X$ and $t \in [0,T]$, $f(., w_t, x) \in B$ and there exists a continuous function $p : [0,T] \to \mathbb{R}_+ = [0,\infty)$ such that

$$\|f(t,\psi,x)\| \leq p(t)(\|\psi\|_C + \|x\|),$$

for every $t \in [0,T]$, $\psi \in C$ and $x \in X$

$(H_2)$ Let $h : [0,T] \times C \to X$ such that for every $w \in B$ and $t \in [0,T]$, $h(., w_t) \in B$ and There exists a continuous function $q : [0,T] \to \mathbb{R}_+$ such that

$$\|h(t,\psi)\| \leq q(t)(\|\psi\|_C),$$

$(H_3)$ Let $g : C^p \to C$ such that there exists positive constants $G$ and $G_1$ satisfying

$$\|(g(x_{t_1}, x_{t_2}, \ldots, x_{t_p}))(t) - (g(y_{t_1}, y_{t_2}, \ldots, y_{t_p}))(t)\| \leq G\|x - y\|_B, \quad t \in [-r,0].$$
$$max \parallel g(x_{t_1}, x_{t_2}, \ldots, x_{t_p}) \parallel \leq G_1$$

$(H_4)$ Let $I_k : X \to X$ are functions such that there exists constants $L_k$ satisfying

$$\|I_k(v)\| \leq L_k\|v\|, \quad v \in X, \quad k = 1, 2, \ldots, m.$$

$(H_5)$ For every positive integer $k$ there exists $\alpha_m \in L^1(0,T)$ such that

$$\sup_{\|\psi\|_C, \|x\| \leq m} \|f(t,\psi,x)\| \leq \alpha_m(t), for \quad t \in [0,T] \quad a.e.$$

$(H_6)$ For each $t \in [0,T]$ the function $f(t, ., .) : C \times X \to X$ is continuous and for each $(\psi, x) \in C \times X$ the function $f(., \psi, x) : [0,T] \to X$ is strongly measurable.

$(H_7)$ For each $t \in [0,T]$ the functions $h(t, .) : C \to X$ are continuous and for each $\psi \in C$ the function $h(., \psi) : [0,T] \to X$ is strongly measurable.

## 3. Existence of mild solution

**Theorem 3.1.** Suppose that the hypotheses $(H_1)$ - $(H_7)$ are satisfied and $\Gamma < 1$, where,

$$\Gamma = KG + K \sum_{0<\tau_k<t} L_k.$$

Then the initial-value problem (1)-(3) has a mild solution $x$ on $[-r, T]$.

*Proof.* We introduce an operator $\mathcal{F}$ on a Banach space B as follows,

$$(\mathcal{F}x)(t) = \begin{cases} \phi(t) - (g(x_{t_1}, \dots, x_{t_p}))(t) \quad \text{if} \quad -r \le t \le 0 \\ T(t)[\phi(0) - g(x_{t_1}, \dots, x_{t_p})(0)] + \int_0^t T(t-s) f\left(s, x_s, \int_0^s k(s,\tau) h(\tau, s_\tau) d\tau\right) ds \\ + \sum_{0<\tau_k<t} T(t-\tau_k) I_k x(\tau_k) \qquad \text{if } t \in (0, T] \end{cases} \tag{4}$$

Let $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$, where,

$$(\mathcal{F}_1 x)(t) = \begin{cases} \phi(t) - (g(x_{t_1}, \dots, x_{t_p}))(t) \quad \text{if} \quad -r \le t \le 0 \\ T(t)[\phi(0) - g(x_{t_1}, \dots, x_{t_p})(0)] + \sum_{0<\tau_k<t} T(t-\tau_k) I_k x(\tau_k) \qquad \text{if } t \in (0, T] \end{cases} \tag{5}$$

and

$$(\mathcal{F}_2 x)(t) = \begin{cases} 0 \quad \text{if} \quad -r \le t \le 0 \\ \int_0^t T(t-s) f\left(s, x_s, \int_0^s k(s,\tau) h(\tau, s_\tau) d\tau\right) ds \qquad \text{if } t \in (0, T] \end{cases} \tag{6}$$

It is easy to see that $\mathcal{F}_1, \mathcal{F}_2 : B \to B$.

Now we will show that $\mathcal{F}_1$ is a contraction on $B$. Let $x, y \in B$. Then for $t \in [-r, 0]$,

$$\|(\mathcal{F}_1 x)(t) - (\mathcal{F}_1 y)(t)\| = \|g(x_{t_1}, \dots, x_{t_p})(t) - g(y_{t_1}, \dots, y_{t_p})(t)\| \le G\|x - y\|_B \tag{7}$$

and for $t \in (0, T]$,

$$\begin{aligned}
\|(\mathcal{F}_1 x)(t) - (\mathcal{F}_1 y)(t)\| &\le \|T(t)\big[g(x_{t_1}, \dots, x_{t_p})(0) - g(y_{t_1}, \dots, y_{t_p})(0)\big] \\
&+ \sum_{0<\tau_k<t} T(t-\tau_k)[I_k x(\tau_k) - I_k y(\tau_k)]\| \\
&\le \|T(t)\| \|g(x_{t_1}, \dots, x_{t_p})(0) - g(y_{t_1}, \dots, y_{t_p})(0)\| \\
&+ \sum_{0<\tau_k<t} \|T(t-\tau_k)\| \|[I_k x(\tau_k) - I_k y(\tau_k)]\| \\
&\le KG\|x-y\|_B + K \sum_{0<\tau_k<t} L_k \|x(\tau_k) - y(\tau_k)\| \\
&\le KG\|x-y\|_B + K \sum_{0<\tau_k<t} L_k \|x-y\|_B \qquad (8)
\end{aligned}$$

As $k \ge 1$, in view of inequality (7) and (8), we can say that inequality (8) holds good for $t \in [-r, T]$. Therefore, for $t \in [-r, T]$,

$$\|(\mathcal{F}_1 x)(t) - (\mathcal{F}_1 y)(t)\| \le \Big(KG + K \sum_{0<\tau_k<t} L_k\Big) \|x-y\|_B$$

which implies

$$\|\mathcal{F}_1 x - \mathcal{F}_2 y\|_B \le \Gamma \|x-y\|_B,$$

where

$$\Gamma = KG + K \sum_{0<\tau_k<t} L_k$$

Since $\Gamma < 1$, the operator $\mathcal{F}_1$ is a contraction on $B$.

First we prove that $\mathcal{F}_2 : B \to B$ is continuous. Let $\{x_n\}$ be a sequence of elements of $B$ converging to $x$ in $B$. Then there exists an integer $N$ such that $\|x_n(t)\| \le N$ for all $n$ and $t \in (0, T]$. So $x_n \in \{x \in B : \|x\|_B \le N\}$ and $x \in \{x \in B : \|x\|_B \le N\}$. Then by using hypothesis $(H_5)$ - $(H_7)$, we have

$$f(t, x_{n_t}, \int_0^t k(t,s)h(s, x_{n_s})ds) \to f(t, x_t, \int_0^t k(t,s)h(s, x_s)ds)$$

for each $t \in (0, T]$ . Since

$$\|f(t, x_{n_t}, \int_0^t k(t,s)h(s, x_{n_s})ds) - f(t, x_t, \int_0^t k(t,s)h(s, x_s)ds)\| \le 2\alpha_{(N_1)'}(t)$$

where $(N_1)' = \max\{N, NLM^*T\}$. Then by dominated convergence theorem, we have

$$\begin{aligned}\|(\mathcal{F}_2x_n)(t) - (\mathcal{F}_2x)(t)\| \leq \int_0^t & \|T(t-s)[f(s, x_{n_s}, \int_0^s k(s,\tau)h(\tau, x_{n_\tau})d\tau) \\ & - f(s, x_s, \int_0^s k(s,\tau)h(\tau, x_\tau)d\tau)]\|ds \\ & \to 0 \quad \text{as} \quad n \to \infty, \forall \quad t \in (0, T].\end{aligned}$$

Since,

$$\|\mathcal{F}_2x_n - \mathcal{F}_2x\|_B = \sup_{t\in[-r,T]} \|(\mathcal{F}_2x_n)(t) - (\mathcal{F}_2x)(t)\|,$$

it follows that

$$\|\mathcal{F}_2x_n - \mathcal{F}_2x\|_B \to 0$$

as $n \to \infty$ which implies $\mathcal{F}_2x_n \to \mathcal{F}_2x$ in $B$ as $x_n \to x$ in $B$. Therefore, $\mathcal{F}_2$ is continuous.

Next we prove that $\mathcal{F}_2$ is completely continuous, that is it maps a bounded set of $B$ into a relatively compact set of $B$. Let $B_m = \{x \in B : \|x\|_B \leq m\}$ for $m \geq 1$. We employ Ascoli-Arzela theorem to show that $\mathcal{F}_2(B_m)$ is relatively compact. For this we prove that $\mathcal{F}_2(B_m)$ is uniformly bounded. From equation (6) and using hypotheses $(H_5)$ - $(H_7)$ and the fact that $\|x\|_B \leq m$, $x \in B_m$ implies $\|x_t\|_C \leq m$, $t \in (0, T]$. We obtain

$$\begin{aligned}\|(\mathcal{F}_2x)(t)\| &\leq \int_0^t \|T(t-s)f(s, x_s, \int_0^s k(s,\tau)h(\tau, x_\tau)d\tau)\|ds \\ &\leq K\int_0^t \Big[\|f(s, x_s, \int_0^s k(s,\tau)h(\tau, x_\tau)d\tau)\|\Big]ds \\ &\leq K\int_0^t \alpha_{N'}(s)ds\end{aligned}$$

where $N' = \max\{m, mLM^*T\}$. This implies that the set $\{(\mathcal{F}_2x)(t) : \|x\|_B \leq m, \ -r \leq t \leq T\}$ is uniformly bounded in $X$ and hence $\mathcal{F}_2(B_m)$ is uniformly bounded.

Now we show that $\mathcal{F}_2$ maps $B_m$ into an equicontinuous family of functions with values in $X$. Let $x \in B_m$ and $t_1, t_2 \in [-r, T]$. Then from the equation (6) and using the hypothesis $(H_5)$ we have

Case 1: Suppose $0 \le t_1 \le t_2 \le T$

$$
\begin{aligned}
&(\mathcal{F}_2x)(t_2) - (\mathcal{F}_2x)(t_1)\\
&= \int_0^{t_2} T(t_2-s) f\Big(s, x_s, \int_0^s k(s,\tau)h(\tau,x_\tau)d\tau\Big)ds\\
&\quad - \int_0^{t_1} T(t_1-s) f\Big(s, x_s, \int_0^s k(s,\tau)h(\tau,x_\tau)d\tau\Big)ds\\
&= \int_0^{t_1} [T(t_2-s) - T(t_1-s)] f\Big(s, x_s, \int_0^s k(s,\tau)h(\tau,x_\tau)d\tau\Big)ds\\
&\quad + \int_{t_1}^{t_2} T(t_2-s) f\Big(s, x_s, \int_0^s k(s,\tau)h(\tau,x_\tau)d\tau\Big)ds\\
&\|(\mathcal{F}_2x)(t_2) - (\mathcal{F}_2x)(t_1)\|\\
&\le \int_0^{t_1} \|T(t_2-s) - T(t_1-s)\| \|f\Big(s, x_s, \int_0^s k(s,\tau)h(\tau,x_\tau)d\tau\Big)\|ds\\
&\quad + \int_{t_1}^{t_2} \|T(t_2-s)\| \|f\Big(s, x_s, \int_0^s k(s,\tau)h(\tau,x_\tau)d\tau\Big)\|ds\\
&\le \int_0^{t_1} \|T(t_2-s) - T(t_1-s)\|\alpha_{N'}(s)ds\\
&\quad + \int_{t_1}^{t_2} \|T(t_2-s)\|\alpha_{N'}(s)ds
\end{aligned}
$$

Case 2: Suppose $-r \le t_1 \le 0 \le t_2 \le T$
Then $(\mathcal{F}_2x)(t_1) = 0$ and therefore proceeding as in Case 1,we get

$$
\begin{aligned}
&\|(\mathcal{F}_2x)(t_2) - (\mathcal{F}_2x)(t_1)\|\\
&\le \int_0^{t_2} \|T(t_2-s)\| \|f(s, x_s, \int_0^s k(s,\tau)h(\tau,x_\tau)d\tau\|ds\\
&\le \int_0^{t_2} \|T(t_2-s)\|\alpha_{N'}(s)ds
\end{aligned}
$$

As $t_2 - t_1 \to 0$ implies $t_2 \to t_1$ and $t_1 \le 0, t_2 \ge 0$ implies $t_2 \to 0$.
Case 3: Suppose $-r \le t_1 \le t_2 \le 0$.Then

$$
\|(\mathcal{F}_2x)(t_2) - (\mathcal{F}_2x)(t_1)\| = 0
$$

Therefore, as $t_2 - t_1 \to 0$,the R.H.S. in the cases 1-3, tends to zero. Thus $\mathcal{F}_2$ maps $B_m$ into an equicontinuous family of functions with values in $X$.

We have already shown that $\mathcal{F}_2(B_m)$ is an equicontinuous and uniformly bounded collection. To prove the set $\mathcal{F}_2(B_m)$ is relatively compact in $B$, it is sufficient, by Arzela-Ascoli's argument, to show that $\mathcal{F}_2$ maps $B_m$ into a relatively compact set in X.

Let $0 < t \leq T$ be fixed and $\epsilon$ a real number satisfying $0 < \epsilon < t$. Moreover for $x \in B_m$, we define

$$(F_{2\epsilon}x)(t) = \int_0^{t-\epsilon} T(t-s)f(s, x_s, \int_0^s k(s,\tau)h(\tau, x_\tau)d\tau)ds$$
$$= T(\epsilon)\int_0^{t-\epsilon} T(t-\epsilon-s)f(s, x_s, \int_0^s k(s,\tau)h(\tau, x_\tau)d\tau)ds$$

Since $T(t)$ is the compact operator ,the set $X_\epsilon(t) = \{(F_{2\epsilon}x)(t) : x \in B_m\}$ is relatively compact in $X$ for every $\epsilon$, $0 < \epsilon < t$. Moreover for every $x \in B_m$, we have

$$\begin{aligned}
&(\mathcal{F}_2x)(t) - (F_{2\epsilon}x)(t) \\
&\leq \int_{t-\epsilon}^t \|T(t-s)f(s, x_s, \int_0^s k(s,\tau)h(\tau, x_\tau)d\tau\|ds \\
&\leq K\int_{t-\epsilon}^t \|f(s, x_s, \int_0^s k(s,\tau)h(\tau, x_\tau)d\tau\|ds \\
&\leq K\int_{t-\epsilon}^t \alpha_{N'}(s)ds
\end{aligned}$$

This shows that there exist relatively compact sets arbitrarily close to the set $\{(\mathcal{F}_2x)(t) : x \in B_m\}$. Hence the set $\{(\mathcal{F}_2x)(t) : x \in B_m\}$ is relatively compact in $X$. This proves that $\mathcal{F}_2$ is completely continuous.

To apply the Krasnoselskii-Schaefer theorem, it remains to show that the set $\Omega(\mathcal{F}) = \left\{x(.) : \lambda\mathcal{F}_1\left(\frac{x}{\lambda}\right) + \lambda\mathcal{F}_2(x) = x\right\}$ is bounded for $\lambda \in (0,1)$.

We denote $M(t) = \sup\{p(t), Lq(t)\}$, $t \in [0,T]$ and $M^* = \sup\{M(t) : t \in [0,T]\}$. $\Lambda_1 = K(\|\phi\|_C + G_1)$ and $L_k{}^* = \frac{L_k}{|\lambda|}$.

To this end let $x(.) \in \Omega(F)$. Then $\lambda\mathcal{F}_1\left(\frac{x}{\lambda}\right) + \lambda\mathcal{F}_2(x) = x$ for some $\lambda \in (0,1)$ and

$$\begin{aligned}
\|x(t)\| &= |\lambda|\|\Big(\mathcal{F}_1\left(\frac{x}{\lambda}\right)\Big)(t) + (\mathcal{F}_2x)(t)\| \\
&\leq \|\mathcal{F}_1\left(\frac{x}{\lambda}\right) + \mathcal{F}_2x(t)\|
\end{aligned}$$

For $t \in [-r, 0]$, we have

$$\|x(t)\| = \|\phi(t) - g\left(\left(\frac{x}{\lambda}\right)_{t_1}, \ldots, \left(\frac{x}{\lambda}\right)_{t_p}\right)(t)\| \leq \|\phi\|_C + G_1 \tag{9}$$

and for $t \in (0, T]$,

$$\begin{aligned}
\|x(t)\| &\leq \|T(t)\|\|\phi(0) - g\left(\left(\frac{x}{\lambda}\right)_{t_1}, \ldots, \left(\frac{x}{\lambda}\right)_{t_p}\right)(0)\| + \sum_{0<\tau_k<t} \|T(t-\tau_k)\|\|I_k\left(\frac{x}{\lambda}\right)(\tau_k)\| \\
&\quad + \int_0^t \|T(t-s)f(s, x_s, \int_0^s k(s,\tau)h(\tau, x_\tau)d\tau\|ds \\
&\leq K(\|\phi\|_C + G_1) + K\sum_{0<\tau_k<t} L_k{}^*\|x(\tau_k)\| + K\int_0^t p(s)[\|x_s\|_C \\
&\quad + L\int_0^s q(\tau)\|x_\tau\|_C d\tau]ds \\
&\leq K(\|\phi\|_C + G_1) + K\sum_{0<\tau_k<t} L_k{}^*\|x(\tau_k)\| + K(M^*)^2T^2\|x\|_B \\
&\quad + \int_0^t KM(s)\|x_s\|_C ds \\
&\leq [\Lambda_1 + \Lambda_2\|x\|_B] + \int_0^t KM(s)\|x_s\|_C ds + \sum_{0<\tau_k<t} KL_k{}^*\|x(\tau_k)\| \qquad (10)
\end{aligned}$$

In view of inequality (9) and (10), we have, for $t \in [-r, T]$,

$$\|x(t)\| \leq [\Lambda_1 + \Lambda_2\|x\|_B] + \int_0^t KM(s)\|x_s\|_C ds + \sum_{0<\tau_k<t} KL_k{}^*\|x(\tau_k)\|$$

Now applying lemma 2.3, we get

$$\|x(t)\| \leq [\Lambda_1 + \Lambda_2\|x\|_B] \prod_{0<\tau_k<t} (1 + KL_k{}^*)exp(KM^*T)$$

By taking supremum over $t \in [-r, T]$, we get,

$$\|x\|_B \leq \frac{\Lambda_1 \prod_{0<\tau_k<t} (1 + KL_k{}^*)exp(KM^*T)}{\left[1 - \Lambda_2 \prod_{0<\tau_k<t} (1 + KL_k{}^*)exp(KM^*T)\right]} = Q, \quad constant$$

This implies the set

$$\Omega(\mathcal{F}) = \left\{x(\,\cdot\,) : \lambda\mathcal{F}_1\left(\frac{x}{\lambda}\right) + \lambda\mathcal{F}_2(x) = x\right\}$$

is bounded for $\lambda \in (0, 1)$. hence by Krasnoselskii-Schaefer fixed point theorem, $\mathcal{F}$ has a fixed point and which is the required mild solution of equations (1)-(3). ■

## 4. Application

To illustrate the application of our result proved in section 3, consider the following semilinear partial functional differential equation of the form

$$\frac{\partial}{\partial t}w(u,t) = \frac{\partial^2}{\partial u^2}w(u,t) + H\left(t, w(u,t-r), \int_0^t k(t,s)P(s,w(s-r))ds\right), 0 \le u \le \pi, t \in [0,T] \tag{11}$$

$$w(0,t) = w(\pi,t) = 0, \quad 0 \le t \le T, \tag{12}$$

$$w(u,t) + \sum_{i=1}^{p} w(u,t_i+t) = \phi(u,t), \quad 0 \le u \le \pi, \quad -r \le t \le 0, \tag{13}$$

$$\Delta w(u,\tau_k) = I_k(w(u,\tau_k)), \quad k = 1,2,\ldots,m. \tag{14}$$

where $0 < t_1 \le t_2 \le t_p \le T$, the functions $H : [0,T]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $P : [0,T]\times\mathbb{R} \to \mathbb{R}$, $I_k : \mathbb{R} \to \mathbb{R}$ are continuous . We assume that the functions $H$, $P$ and $I_k$ satisfy the following conditions:
For every $t \in [0,T]$ and $u, x \in \mathbb{R}$, there exists nondecreasing continuous functions $p, q$, $c_k$ and $d$ constants such that

$$|H(t,u,x)| \le p(t)(|u| + |x|)$$

$$|P(t,u)| \le q(t)(|u|)$$

$$|I_k(x)| \le c_k|x|, \quad k = 1,2,\ldots,m$$

$$\sum_{i=1}^{p} |w(u,t_i+t)| \le d$$

and $Kd + K \sum_{0<\tau_k<t} c_k < 1$.

Let us take $X = L^2[0,\pi]$. Define the operator $A : X \to X$ by $Az = z''$ with domain $D(A) = \{z \in X : z,\ z'$ are absolutely continuous, $z'' \in X$ and $z(0) = z(\pi) = 0\}$. Then the operator $A$ can be written as

$$Az = \sum_{n=1}^{\infty} -n^2(z,z_n)z_n, \ \ z \in D(A)$$

where $z_n(u) = (\sqrt{2/\pi})\sin nu,\ n = 1,2,\ldots$ is the orthogonal set of eigenvectors of $A$ and $A$ is the infinitesimal generator of an analytic semigroup $T(t),\ t \ge 0$ and is given by

$$T(t)z = \sum_{n=1}^{\infty} exp(-n^2t)(z,z_n)z_n, \ \ z \in X.$$

Now, the analytic semigroup $T(t)$ being compact, there exists constant $K$ such that

$$|T(t)| \leq K, \quad \text{for each } t \in [0,T].$$

Define the functions $f : [0,T] \times C \times X \to X$,$h : [0,T] \times C \to X$, $I_k : X \to X$ as follows

$$f(t,\psi,x)(u) = H(t,\psi(-r)u, x(u)),$$
$$h(t,\phi)(u) = P(t,\phi(-r)u)$$

for $t \in [0,T], \psi,\phi \in C$, $x \in X$ and $0 \leq u \leq \pi$. With these choices of the functions the equations (11)-(14) can be formulated as an abstract integro-differential equation in Banach space $X$:

$$x'(t) = Ax(t) + f\big(t, x_t, \int_0^t k(t,s)h(s,x_s)ds\big), \quad t \in [0,T]$$
$$x(t) + (g(x_{t_1},\ldots,x_{t_p}))(t) = \phi(t), \quad t \in [-r,0]$$
$$\Delta x(\tau_k) = I_k x(\tau_k), \quad k = 1,2,\ldots,m,$$

Since all the hypotheses of the theorem 3.1 are satisfied, the Theorem 3.1, can be applied to guarantee the existence of mild solution $w(u,t) = x(t)u, \quad t \in [0,T], u \in [0,\pi]$, of the semilinear partial integro-differential equation (11)-(14).

## References


[1] H. Acka, A. Boucherif and Valery Covachev, Impulsive functional differential equations with nonlocal conditions, *IJMMS*, 29:5 (2002), pp: 251–256.

[2] T. A. Burton, Volterra integral and differential equations, *Academic Press, New York*, 1983.

[3] T. A. Burton and C. Kirk, A fixed point theorem of Krasnoselsii-Schaefer type, *Math. Naehr.*, 189(1998), 23–31.

[4] L. Byszewski, On mild solution of a semilinear functional differential evolution nonlocal problem, *J. of Appl.maths and Stoch. Anal.*, Vol.10(3), 1997, pp. 265–271.

[5] L. Byszewski and H. Akca, Existence of solutions of a semilinear functional evolution nonlocal problem, *Nonlinear Anal.*, 34(1998), 65–72.

[6] L. Byszewski and V. Lakshamikantham, Theorems about the existence and uniqueness of a solution of a nonlocal abstract cauchy problem in a Banach space, *Applicable Anal.*, 40(1990), pp. 11–19.

[7] J.K. Hale, Theory of functional differential equations, *Springer*, 1977.

[8] R. S. Jain and M. B. Dhakne, On global existence of solutions for abstract nonlinear functional integro-differential equations with nonlocal condition, *Contempoary Math. and stat.*, Vol.1 (2013), pp. 44–53.

[9] Rupali S. Jain and M. B. Dhakne, On mild solutions of nonlocal semilinear functional integro-differential equations, *Malaya Journal of Mathematik*, 3(1) (2013), 27–33.

[10] V. Lakshmikantham, D.D. Bainov and P.S. Simeonov, Thoery of Impulsive Differential Equations, *World Scientific*, Singapore, 1989.

[11] Lin Y. and Liu J.H., Semilinear integrodifferential equations with nonlocal Cauchy problem, *Nonlinear Analysis: Theory, Meth. and Appl.*, 26(1996), pp. 1023–1033.

[12] S. K. Ntouyas, Initial and boundary value problems for functional differential equations via the topological transversality method: A survey, *Bull. Greek Math. Soc.*, 1998.

[13] S. K. Ntouyas, Nonlocal initial and boundary value problems: A survey, *Handbook of differential equations: Ordinary differential equations*, Vol.2, 2006, PP. 461–557.

[14] A. Pazy, Semigroups of linear operators and applications to partial differential equations, *Springer Verlag, New York*, 1983.

[15] A. M. Samoilenko and N.A. Perestyuk, Impulsive Differential Equations, *World Scientific*, Singapore, 1995.

[16] Shao chun Ji and Shu Wen, Nonlocal Cauchy problem for impulsive differential equations in Banach spaces, *Int. J. of Nonl. Sci.*, Vol 10(2010), No. 1, pp: 88–95.