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Representation of Stochastic Process by Means of Stochastic Integrals

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Abstract

This paper produces an affirmative solution to Kolmogorov concerning the normal distribution.

Keywords: Stochastic integral, Kolmogorov problem.

1.Introduction

Let $X(t)$ be a stochastic process for $t \geq 0$; the random variable $X(t_2) - X(t_1)$ is called the increment of the process $X(t)$ over the interval $[t_1, t_2]$. A process $X(t)$ is said to be homogeneous if the distribution function of the increment $X(t + \tau) - X(t)$ depends only on the length τ of the interval but is independent of t. Two intervals are said to be non – overlapping intervals if they have no common interior point. A process $X(t)$ is called a process with independent increments if the increments over non – overlapping intervals are independent. A process is said to be continuous at the point t if for any > 0 $\lim_{\tau \to 0} P(|X(t + \tau) - X(t)| > \varepsilon) = 0$. a process is continuous in an interval $[A, B]$ if it is continuous in every point of $[A, B]$.

Let $X(t)$ be a homogeneous and continuous process with independent increments and let us denote the characteristic function of the increment $X(t + \tau) - X(t)$ by $f(u, \tau)$. It is known that $f(u, \tau)$ is infinitely divisible and that $f(u, \tau) = f(u, 1)$.

Let b be a function defined in $[A, B]$ and v a non – negative function in $[A, B]$. Let us consider for each integer n a subdivision of $[A, B]$ $(\mathcal{D}_n)$ $A = t_{n,0} < t_{n,1} < \cdots \ldots \ldots \ldots < t_{n,n} = B$ $\quad (n = 1,2, \ldots \ldots \ldots)$ and assume that

$$\operatorname*{Max}_{1 \leq k \leq n} \left(t_{n,k} - t_{n,k-1}\right) \to 0 \qquad as\ n \to \infty$$

In each subinterval $[t_{n,k-1}, t_{n,k}]$ select a point $t^*_{n,k}$ $(k = 1,2, \ldots\ldots, n)$. Let us form the sums

$$S_n = \sum_{k=1}^{n} b\left(t^*_{n,k}\right)\left[X\left(v(t_{n,k})\right) - X\left(v(t_{n,k-1})\right)\right]$$

If the sequence S_n converges in probability to a random variable S, and is this limit is independent of the choice of the subdivision $(\mathcal{D}_n)$ and the points $t^*_{n,k}$, then say that S is a stochastic integral in the sense of convergence in probability and write

$$S = \int_A^B b(t)dX\big(v(t)\big).$$

The following theorem gives a condition ensuring the existence of the stochastic integral in the sense of convergence in probability.

Theorem 1.1
Let $X(t)$ be a homogeneous and continuous process with independent increments defined for $t \geq 0$. Suppose that the function b is continuous in $[A, B]$ and v is a non – decreasing, non – negative and left continuous function in $[A, B]$. Then the stochastic integral

$$\int_A^B b(t)dX\big(v(t)\big) \tag{1.1}$$

exists in the sense of convergence in probability and its characteristic function h is given by

$$\log h(u) = \int_A^B log\, f(u \cdot b(t))dv(t) \tag{1.2}$$

Let b and v be as in theorem 1.1. It is well known that there exists a finite Borel measure V with support contained in $[A, B]$ such that

$$V((-\infty, t)) = \begin{cases} 0 & if\ t < A \\ v(t) - v(A) & if\ A \leq t \leq B \\ v(B) - v(A) & if\ t > B \end{cases}$$

Put

$$v_b(t) = V(b^{-1}(-\infty, t))$$

then v_b is a non – decreasing, non – negative and left continuous function. Further we put,

$$C = \min_{A \leq t \leq B} b(t)$$

and

$$D = \max_{A \leq t \leq B} b(t)$$

By theorem 1.1 the stochastic integral

$$\int_A^B t dX(v_b(t)) \tag{1.3}$$

exists and its characteristic function h_1 is given by

$$\log h_1(u) = \int_A^B \log f(u \cdot b(t)) dv_b(t)$$

by the transform formula for integrals, we have

$$\int_A^B \log f(u \cdot b(t)) dv(t) = \int_A^B \log f(u \cdot t) dv_b(t).$$

Theorem 1.2
Let $X(t), b$ and v be as in Theorem 1.1. Then the integrals (1.1) and (1.3) are identically distributed.

2. Representation Theorem

Theorem 2.1
Let $X(t)$ be a homogeneous and continuous process with independent increments defined for $t \geq 0$ and let the Levy canonical representation of the characteristic function of $X(0) - X(1)$ be given by a, σ, M and N. Let v be a non – decreasing , non negative and left continuous function in $[A, B]$. Then the Levy canonical representation for the characteristic function of the stochastic integral

$$\int_A^B t dX(v(t)) \tag{2.1}$$

is given by the following formulas:

$$a_v = \int_A^B \left(ta + t(1-t^2)\right) \int_{0+}^{\infty} \frac{x^3}{(1+(tx)^2)(1+x^2)} \, d(M(-x) + N(x)) + dv(t); \tag{2.2}$$

$$\sigma_v^2 = \sigma^2 \int_A^B t^2 dv(t) \tag{2.3}$$

$$M_v(x) = \int_{\min(A,0)}^{\min(B,0)} -N\left(\frac{x}{t}\right) dv(t) + \int_{\max(A,0)}^{\max(B,0)} M\left(\frac{x}{t}\right) dv(t) \qquad (x < 0) \tag{2.4}$$

$$N_v(x) = \int_{\min(A,0)}^{\min(B,0)} -M\left(\frac{x}{t}\right) dv(t) + \int_{\max(A,0)}^{\max(B,0)} N\left(\frac{x}{t}\right) dv(t) \qquad (x > 0) \tag{2.5}$$

Lemma 2.1
The function g is an infinitely divisible characteristic function if, and only if, it can be

written in the form

$$\log g(u) = iau + \frac{\sigma^2}{2} u^2 + \int_{-\infty}^{-0} r(u,x) dM(x) + \int_{+0}^{\infty} r(u,x)\, dN(x)$$

where a, σ are real constants; M and n are non – decreasing in the intervals $(-\infty, 0)$ and $(0, \infty)$ respectively, with

$$M(-\infty) = N(\infty) = 0$$

$$\int_{-\in}^{0-} x^2 dM(x) < \infty \quad and \quad \int_{0+}^{\in} x^2 dN(x) < \infty \quad for\ every\ \varepsilon > 0$$

and

$$r(u,x) = e^{iux} - 1 - (iux/(1 + x^2)) \tag{2.6}$$

Proof of theorem 2.1

With out loss of generality let us assume that $A \leq 0 \leq B$. First we assume that there exists a number $t_0 > 0$ such that t_0 is a point of continuity of υ and

$$\upsilon(t_0) - \upsilon(-t_0) = 0 \tag{2.7}$$

The characteristic function of (2.1) is denoted by h. Then by theorem 1.1 we have

$$\log h(u) = \int_A^B log\ f(u \cdot t) d\upsilon(t) \tag{2.8}$$

Now let us define a function s by

$$s(u,x,t) = r(ut,x) - r(u,tx)$$

where in view of (2.6)

$$s(u,x,t) = \frac{it(1-t^2)x^3 u}{(1+(tx)^2)(1+x^2)}$$

Since $s(u,x,t) = o(x^2)$ as $x \to 0$ and $s(u,x,t) = o(1)$ as $x \to \infty$ the function s is integrable with respect to M and N. By Lemma 2.1 and the definition of s we have,

$$\log f(ut) = iaut - \frac{\sigma^2}{2} (ut)^2 + \int_{-\infty}^{-0} (r(u,tx) + s(u,x,t)) dM(x)$$

$$+ \int_{+0}^{\infty} \left(r(u,tx) + s(u,x,t)\right) dN(x) \tag{2.9}$$

By virtue of (2.8) and (2.9) let us obtain

$$\log h(u) = iau \int_A^B t d\upsilon(t) - \frac{\sigma^2}{2} u^2 \int_A^B t^2 d\upsilon(t)$$

$$+\int_A^B \left(\int_{-\infty}^{-0} s(u,x,t)dM(x) + \int_{+0}^{\infty} s(u,x,t)\, dN(x) \right) dv(t)$$

$$+\int_A^B \left(\int_{-\infty}^{-0} r(u,tx)dM(x) + \int_{+0}^{\infty} r(u,tx)\, dN(x) \right) dv(t)$$

Using the definitions of a_v and σ_v we can write this relation in the form

$$\log h(u) = ia_v t - \frac{\sigma_v^2}{2}(ut)^2 + \int_A^B \int_{-\infty}^{-0} r(u,tx)dM(x)dv(t)$$

$$+\int_A^B \int_{0+}^{\infty} r(u,tx)dN(x)dv(t) \tag{2.10}$$

and in view of (2.7), Let us have

$$\log h(u) = ia_v t - \frac{\sigma_v^2}{2}(ut)^2 + \int_A^{-t_0} \int_{-\infty}^{-0} r(u,tx)dM(x)dv(t)$$

$$+\int_{t_0}^B \int_{-\infty}^{-0} r(u,tx)dM(x)dv(t)$$

$$+\int_A^{t_0} \int_{0+}^{\infty} r(u,tx)dN(x)dv(t)$$

$$+\int_{t_0}^B \int_{0+}^{\infty} r(u,tx)dN(x)dv(t) \tag{2.11}$$

Decomposing the third term on the right – hand side of (2.11) we get for every $\varepsilon > 0$.

$$I = \int_A^{-t_0} \int_{-\infty}^{-0} r(u,tx)dM(x)dv(t)$$

$$= \int_A^{-t_0} \int_{-\epsilon}^{-0} r(u,tx)dM(x)dv(t) + + \int_A^{t_0} \int_{-\infty}^{-\epsilon} r(u,tx)dN(x)dv(t)$$

$$= I_1 + I_2$$

Applying L' Hospital's rule twice we find

$$\lim_{x\to\infty} \frac{r(u,tx)}{x^2} = -\frac{(ut)^2}{2}$$

Hence there is a constant C_1 such that for fixed u and $t \in [A, -t_0]$

$$|r(u,tx)| \le C_1(ut)^2x^2$$

Therefore let us get for I_1 the estimation $(\varepsilon \to 0+)$

$$|I_1| \le C_1u^2 \int_A^{-t_0} t^2 dv(t) \int_{-\in}^{0-} x^2\, dM(x) = o(1)$$

Further we can transform I_2 in the following way.

$$I_2 = \int_A^{-t_0} \int_{\in}^{\infty} r(u,x) d_x\left(-M\left(\frac{x}{t}\right)\right) dv(t)$$

$$= \int_{\in}^{\infty} r(u,x) d_x \int_A^{-t_0} -M\left(\frac{x}{t}\right) dv(t)$$

so let us obtain as $\varepsilon \to 0$

$$I = \int_{0+}^{\infty} r(u,x) d\left(\int_A^{-t_0} -M\left(\frac{x}{t}\right) dv(t)\right)$$

Transforming the fourth, fifth and sixth terms of (2.11) in a similar manner to the third one we get

$$\log h(u) = ia_v t - \frac{\sigma_v^2}{2}\,(ut)^2 + \int_{0+}^{\infty} r(u,x) d\left(\int_A^{-t_0} -M\left(\frac{x}{t}\right) dv(t)\right)$$

$$+ \int_{-\infty}^{0-} r(u,x) d \int_{t_0}^{B} M\left(\frac{x}{t}\right) dv(t)$$

$$+ \int_{-\infty}^{0-} r(u,x) d \int_A^{-t_0} -N\left(\frac{x}{t}\right) dv(t)$$

$$+ \int_{0+}^{\infty} r(u,x) d \int_{t_0}^{B} N\left(\frac{x}{t}\right) dv(t)$$

Finally, using the definition of M_v and N_v , we can rewrite this relation in the form,

$$\log h(u) = ia_{\upsilon}t - \frac{\sigma_{\upsilon}^2}{2}\,(ut)^2 + \int_{-\infty}^{0-} r(u,x)d\,M_{\upsilon}(x) + \int_{0+}^{\infty} r(u,x)d\,N_{\upsilon}(x)$$

Let us complete the proof by showing that $a_{\upsilon}, \sigma_{\upsilon}, M_{\upsilon}$ and N_{υ} satisfy the condition of Lemma 2.1. Obviously a_{υ} and σ_{υ}^2 are real constants and $\sigma_{\upsilon}^2 \geq 0$. By definition it is easily seen that M_{υ} and N_{υ} are non – decreasing in the intervals $(-\infty, 0)$ and $(0,\infty)$, respectively, having the properties
$M_{\upsilon}(-\infty) = N_{\upsilon}(\infty) = 0$

For every $\varepsilon > 0$ we obtain the inequality

$$\int_{0+}^{\in} x^2 dN_{\upsilon}(x) = \int_{t_0}^{B} t^2 \int_{0+}^{\in/t} x^2 dN(x)d\upsilon(t) + \int_{A}^{-t_0} t^2 \int_{\in/t}^{0-} x^2 dM(x)d\upsilon(t)$$

$$\leq \int_{t_0}^{B} t^2 d\upsilon(t) \int_{0+}^{\in/t_0} x^2 dN(x) + \int_{A}^{-t_0} t^2\, d\upsilon(t) \int_{\in/t_0}^{0-} x^2 dM(x) < \infty$$

Analogously, let us get

$$\int_{-\in}^{0-} x^2 dM_{\upsilon}(x) < \infty$$

Then Lemma 2.1 shows he statement provided that (2.7) is valid.

Now let us turning to the general case n which (2.7) need not be true. Put for $n \geq \max(-1/A, 1/B)$

$$\upsilon_n(t) = \begin{cases} \upsilon(t) & if\ t \leq -1/n \\ \upsilon(-1/n) & if\ -1/n < t \leq 1/n \\ \upsilon(t) & if\ t > 1/n \end{cases}$$

Obviously , we have

$$\lim_{n\to\infty} \upsilon_n(t) = \upsilon(t) \tag{2.12}$$

and the functions υ_n satisfying (2.7) are non – decreasing, non – negative and left – continuous. Hence let us can apply the first part of the proof to the stochastic integrals

$$\int_{A}^{B} t dX\big(\upsilon_n(t)\big) \tag{2.13}$$

and obtain representation of $a_{\upsilon_n}, \sigma_{\upsilon_n}, M_{\upsilon_n}$ and N_{υ_n} by formulas analogous to (2.2) – (2.5) using Helly's second theorem, we get

$$\lim_{n\to\infty}\int_A^B log\, f(u\cdot t)dv(t) = \int_A^B log\, f(u\cdot t)dv(t) \quad (2.14)$$

Because $log\, f(u\cdot t)$ – considered as function of t – is continuous and bounded and by (2.12) the sequence v_n converges weakly to v. Let us denote the characteristic function of (2.13) by h_v. In view of theorem 1.1 . relation (2.14) is equivalent

$$\lim_{n\to\infty} h_n(u) = h(u)$$

Using the known fact that under this circumstance $a_v \to a_{v_n}, \sigma_v \to \sigma_{v_n}, M_{v_n} \Rightarrow M_v$ and $N_{v_n} \Rightarrow N_v$ ($\Rightarrow$ stands for weak convergence) the statement follows.

3.ON A PROBLEM OF KOLMOGOROV CONCERNING THE NORMAL DISTRIBUTION

The following theorem gives an affirmative answer to a question posed by A. N. Kolmogorov.

Theorem3.1

Let F be an infinitely divisible (i.d.) distribution function and

$$F\ (x) == \Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{u^2/2}\, du \text{ for } x < 0\text{; then } F\ (x) \equiv \Phi(x)$$

For the proof let us start by collecting together several lemmas.

Lemma 3.2

Let F be i. d. and

$$\int_{-\infty}^{\infty} e^{-yx}\, dF(x) < \infty, \quad y > 0.$$

Then the Levy representation of $f(t) = \int_{-\infty}^{\infty} e^{itx}\, dF(x)$

$$\log f(iy) = -ay + \frac{\sigma^2}{2}y^2 + \int_{-\infty}^{-0} r(y,u)dM(u) + \int_{+0}^{\infty} r(y,u)\, dN(u)$$

where

$$r(y,u) = e^{-yu} - 1 + \frac{yu}{1+u^2}$$

holds for $y > 0$.

Lemma 3.3

Under the assumptions of lemma 3.2 $log\, f(iy)$ increases at least exponentially, if $M(U) \not\equiv 0$.

Proof
Using the formula

$$r(y,u) = e^{-yu} - 1 + yu - y\frac{u^3}{1+u^2}$$

we obtain for $\eta > 0$

$$\int_{+0}^{\infty} r(y,u)\,dN(u) = y\int_{+\infty}^{\eta} l(yu)u\,dN(u) - y\int_{+0}^{\eta}\frac{u^3}{1+u^2}dN(u) + \int_{\eta}^{\infty} r(y,u)\,dN(u)$$

where

$$l(v) = v^{-1}(e^{-v} - 1 + v) > 0$$

whence we get the estimation

$$\int_{+0}^{\infty} r(y,u)\,dN(u) \geq -y\int_{+0}^{\eta}\frac{u^3}{1+u^2}dN(u) - \int_{\eta}^{\infty} dN(u) = L_\eta(y)\ (say)$$

Now we choose $-\infty < p < q < 0$ and assume $M(q) > M(p)$. Then we obtain from lemma 3.2

$$\log f(iy) \geq -ay + \frac{\sigma^2}{2}y^2 + \int_p^q\left(e^{y|u|} - 1 - \frac{y|u|}{1+u^2}\right)dM(u) + L_\eta(y)$$

This proves the assertion.

Lemma 3.4
Let

$$\Phi_-(x) = \begin{cases} 2\Phi(x), & x < 0 \\ 1, & x > 0 \end{cases}$$

then the corresponding characteristic function (c. f.) is

$$\varphi_-(t) = e^{-t^2/2}\left(1 - i\sqrt{\frac{2}{\pi}}\int_0^t e^{w^2/2}\,dw\right)$$

with the asymptotic behaviour

$$\varphi_-(iy) = e^{-y^2/2}\left(2 + o\left(e^{-y^2/2}\right)\right)\ (y \to \infty).$$

Proof
We have to calculate

$$f(t) = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{0} e^{itu} e^{-u^2/2}\ du = \sqrt{\frac{2}{\pi}}\ e^{-t^2/2} \int_{-\infty}^{0} e^{-\frac{1}{2}(it-u)^2}\ du$$

The second integral can be evaluated by means of contour integration starting with

$$0 = \int_{Z}^{0} e^{-\frac{1}{2}(it-\varsigma)^2}\ d\varsigma + \int_{0}^{it} (\dots\dots) + \int_{it}^{it+Z} (\dots\dots) + \int_{it+Z}^{Z} (\dots\dots)$$

and letting $Z \to -\infty$. Equation (4) is an easy consequence of (3).

Lemma 3.5

Let $\hat{f}(t)$ be the c. f. of an i. d. distribution $\hat{F}(x)$. Then $\hat{F}(0) = 0, \hat{F}(\varepsilon) > 0$ if and only if in the corresponding Levy representation— characterized by $(\hat{a}, \hat{\sigma}, \widehat{M}, \widehat{N})$.

We have

$$\hat{\sigma}^2 = 0, \widehat{M}(u) \equiv 0 (u < 0),$$

$$\int_{+0}^{1} u d\widehat{N}(u) < \infty, \hat{a} + \int_{+0}^{\infty} \frac{u}{1+u^2} d\widehat{N}(u) = 0$$

In this case we have

$$\log \hat{f}(iy) = \int_{+0}^{\infty} (e^{-yu} - 1)\, d\widehat{N}(u)$$

Proof

Putting $G(x) = 2F(x) - 1, x > 0$, we represent F by

$$F(x) = \frac{1}{2}[\Phi_{-}(x) + G(x)].$$

We denote the c. f. of $G(x)$ by $g(t)$ with the property $g(iy) < 1\ (y > 0)$ and obtain by lemma 3

$$\log f(iy) = \int_{-\infty}^{\infty} e^{-yx}\, dF(x) = \frac{1}{2}[\varphi_{-}(iy) + g(iy)] = e^{y^2/2}\left[1 + O\left(e^{-y^2/2}\right)\right]$$

i.e

$$\log f(iy) = \frac{y^2}{2} + O\left(e^{-y^2/2}\right)$$

By lemma 1 we also have the representation (1). We are going to compare the asymptotic $(y \to \infty)$ behaviour of (1) and (5). From Lemma 2 we immediately get the conclusion $M(u) \equiv 0 (\ u < 0\ )$.

So we have to direct our attention to the function $N\ (u)\ (u\ >\ 0)$, which always has the property

$$\int_{+0}^{1} u^2 dN(u) < \infty.$$

We have to consider two cases.

$$\alpha) \qquad \int_{+0}^{1} udN(u) < \infty.$$

The function $l(u)$ in (2) is increasing, hence, putting

$$H(u) = \int_{1}^{u} vdN(v)(NB\colon H(+0) = -\infty),$$

We have for $\eta > 0$

$$R_\eta(y) =: \int_{0}^{\eta} l(yu)udN(u) > \int_{1}^{y\eta} l(v)dH\left(\frac{v}{y}\right) \geq e^{-1}[H(\eta) - H(y^{-1})]$$

Therefore, by assumption $\alpha)$, $\lim_{y\to\infty} R_\eta(y) = \infty$. Now we conclude from (1) and

$$\log f(iy) = \frac{\sigma^2}{2}\, y^2 + y\left[-a + R_\eta(y) - \int_{+0}^{\eta} \frac{u^3}{1+u^2} dN(u) + \int_{\eta}^{\infty} \frac{u}{1+u^2} dN(u)\right] + \int_{\eta}^{\infty} (e^{-yu} - 1)\, dN(u) = \frac{\sigma^2}{2}\, y^2 + yR_\eta(y)(1 + o(1))$$

This formula must be compared with (5).

Let us first assume that $\lim_{y\to\infty} y^{-1}R_\eta(y) = 0$. Then it follows $\sigma = 1$, and we have the contradiction

$$yR_\eta(y) = O\left(e^{-y^2/2}\right)$$

On the other hand, if we have

$$\lim_{y\to\infty} y^{-1}R_\eta(y) = l_\eta > 0,$$

then we get

$$\frac{\sigma^2}{2} + l_\eta = \frac{1}{2}$$

Now we note that (as $l(v)v^{-1} < \text{c}, \forall\, v > 0$)

$$yR_\eta(y) = y^2 \int_{+0}^{\eta} \left[\frac{l(yu)}{yu}\right] u^2 dN(u) \le cy^2 \int_{+0}^{\eta} u^2\, dN(u),$$

Therefore l_η can be made arbitrarily small.

Hence we again obtain $\sigma = 1$ and the wrong equation (6). So assumption $\alpha)$ is false.

$$\beta) \qquad \int_{+0}^{1} u dN(u) < \infty.$$

In this case (1) yields

$$\log f(iy) = \frac{\sigma^2}{2} y^2 + y\left[-a + \int_{\eta}^{\infty} \frac{u}{1+u^2} dN(u)\right] + S_\eta(y) + \int_{\eta}^{\infty} (e^{-yu} - 1)\, dN(u)$$

where

$$S_\eta(y) = \int_{+0}^{\eta} (e^{-yu} - 1)\, dN(u) = y \int_{+0}^{\eta} \frac{e^{-yu} - 1}{yu} d \int_{+0}^{u} v dN(u)$$

The integrand is increasing, i. e.

$$-y \int_{+0}^{\eta} v dN(u) \le S_\eta(y) \le 0$$

Comparing (7) and (5), we now immediately obtain $\sigma = 1$ and

$$\left[-a + \int_{+0}^{\infty} \frac{u}{1+u^2} dN(u)\right] + \frac{1}{y} S_\eta(y) = O\left(e^{-y^2/2}\right)$$

Putting

$$\lim_{y\to\infty} y^{-1} S_\eta(y) = s_\eta$$

we get from (9)

$$\left[-a + \int_{+0}^{\infty} \frac{u}{1+u^2} dN(u)\right] + S_\eta = 0$$

But (8) shows, that $|S_\eta| \le \int_{+\infty}^{\eta} v dN(v)$ can be chosen arbitrarily small. Therefore

$$\left[-a + \int_{+0}^{\infty} \frac{u}{1-u^2} dN(u)\right] = 0$$

From Lemma 4 and (7) we now conclude that

$$f(iy) = e^{\frac{\sigma^2}{2}y^2}\hat{f}(iy)$$

where $\hat{f}(t)$ is the c. f. of a distribution function $\hat{F}(x)$ with $\hat{F}(0) = 0, \hat{F}(\varepsilon) > 0$.

In view of (5) this is possible only for $\hat{f}(iy) \equiv 1$, i. e. $\widehat{N}(u) = N(u) \equiv 0$. Finally we obtain $a = 0$ from (10). The proof is complete.

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