International Journal of Mathematics Research.
ISSN 0976-5840 Volume 5, Number 5 (2013), pp. 431-440

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On Quasi Weak Commutative Near – Rings

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Abstract:

A right near - ring N is called weak Commutative, (Definition 9.4 Pilz [7]) if $xyz = xzy$ for every $x,y,z \in N$. A right near - ring N is called pseudo commutative (Definition 2.1, S.Uma and others [8]) if $xyz = zyx$ for all $x,y,z \in N$.It is quite natural to investigate the properties of N if $xyz = yxz$ for every $x,y,z \in N$.We call such a near - ring as quasi weak commutative. We obtain some interesting results on Quasi weak commutative near - ring.

1.Introduction:

Throughout this paper, N denotes a right near – ring (N,+, .) with atleast two elements . For any non-empty subset A of N,we denote $A – \{0\} = A^*$ .The following definitions and results are well known.

Definition: 1.1
An element $a \varepsilon N$ is said to be
1. Idempotent if $a^2 = a$.
2. Nilpotent, if there exists a positive integer k such that $a^k = 0$.

Result : 1.2 (Theorem 1.62 Pilz [7])
Each near – ring N is isomorphic to a subdirect product of subdirectly irreducible near – rings.

Definition : 1.3
A near – ring N is said to be zero symmetric if ab = 0 implies ba = 0,where a,b ε N.

Result : 1.4
If N is zero symmetric, then
1. Every left ideal A of N is an N – subgroup of N.
2. every ideal I of N satisfies the condition NIN $\subseteq$ I . (i.e) every ideal is an N – subgroup.
3. $N^* I^* N^* \subseteq I^*$.

Result :1.5
Let N be a near – ring.Then the following are true
1. If A is an ideal of N and B is any subset of N, then (A: B) = { n ϵ N such that nB $\subseteq$ A } is always a left ideal.
2. If A is an ideal of N and B is an N - subgroup, then (A: B) is an ideal. In particular if A and B are ideals of a zero - symmetric near - ring,then (A : B) is an ideal.

Result : 1.6
1. Let N be a regular near - ring, a ϵ N and a = axa, then ax,xa are idempotents and so the set of idempotent elements of N is non – empty.
2. axN = aN and Nxa = Na.
3. N is S and S′ near - rings.

Result : 1.7 (Lemma 4 Dheena [1])
Let N be a zero - symmetric reduced near - ring.For any a,b ϵ N and for any idempotent element e ϵ N, abe = aeb.

Result : 1.8 (Gratzer [4] and Fain [3])
A near - ring N is sub - directly irreducible if and only if the intersection of all non - zero ideals of N is not zero.

Result : 1.9 (Gratzer [4])
Each simple near – ring is sub directly irreducible.

Result : 1.10 (Pilz [7])
A non - zero symmetric near – ring N has I F P if and only if (O : S) is an ideal for any subset S of N.

Result : 1.11 (Oswald [6])
An N - subgroup A of N is essential if A∩ B = {0},where B is any N subgroup of N , implies B = {0}.

Definition : 1.12
A near - ring N is said to be reduced if N has no non-zero nilpotent elements.

Definition : 1.13
A near - ring N is said to be an integral near – ring, if N has no non – zero divisors.

Lemma: 1.14
Let N be a near-ring. If for all a ε N, $a^2 = 0 \Rightarrow a = 0$, then N has no non-zero nilpotent elements.

Definition : 1.15
Let N be a near-ring. N is said to satisfy intersection of factors property (IFP) if ab = 0 => anb = 0 for all n ϵ N, where a,b ϵ N.

Definition : 1.16
1. An ideal I of N is called a prime ideal if for all ideals A,B of N, AB is subset of I => A is subset of I or B is subset of I.
2. I is called a semi-prime ideal if for all ideals A of N, A^2 is subset of I implies A is subset of I.
3. I is called a completely semi – prime ideal, if for any x ϵ N, x^2 ϵ I => x ϵ I.
4. A completely prime ideal, if for any x,y ϵ N, xy ϵ I => x ϵ I or y ϵ I
5. N is said to have strong IFP, if for all ideals I of N, ab ϵ I implies anb ϵ I for all n ϵ N.

Result: 1.17 (Proposition 2.4 [8])
Let N be a Pseudo commutative near – ring. Then every idempotent element is central.

2. Definition and Basic Properties
In this section we introduce the concept of quasi weak commutative near – rings.

Definition :2.1
A near – ring N is said to be quasi weak commutative near – ring if xyz = yxz for all x, y, z ϵ N.

Every commutative near – ring is quasi weak commutative.

Proposition:2.2
Every quasi weak commutative near – ring is zero symmetric.

Proof:
Let N be a quasi weak commutative near – ring.

For every a ϵ N,

a.0 = a(00) = (0a)0 = 00 = 0.

This proves N is zero symmetric.

Proposition:2.3
Let N be both weak commutative and quasi weak commutative right near – ring. Then N is pseudo commutative.

Proof:
For all x, y, z ϵ N, we have

xyz = yxz (N is quasi weak commutative)
= yzx (N is weak commutative)
= zyx (N is quasi weak commutative)

This proves N is pseudo commutative.

Proposition:2.4
Homomorphic image of a quasi weak commutative near – ring is also right pseudo commutative near – ring.

Proof:
Let N be a quasi weak commutative near – ring and f: N → M be an Endomorphism of near – rings.

For all x, y, z ϵ N.

f(x)f(y)f(z) = f(xyz)
= f(yxz)
= f(y)f(x)f(z)

This proves M is quasi weak commutative.

Corollary:2.5
Let N be a quasi weak commutative near – ring. If I is any ideal of N, then N/I is also quasi commutative.

Theorem:2.6
Every quasi weak commutative near – ring N is isomorphic to a sub-direct product of sub-directly irreducible quasi weak commutative near – rings.

Proof:
By Result 1.2, N is isomorphic to a sub-direct product of sub-directly irreducible near – rings N_k, and each N_k is a homomorphic image of N, under the projection map π_k : N → N_k. The desired result follows from Proposition 2.7.

Proposition:2.7
Any weak commutative near ring with left identity is a quasi weak commutative near – ring.

Proof:
Let $e \in N$ be a left identity.

For any $a,b,c \in N$

$$\begin{aligned} abc &= e(abc) \\ &= (eab)c \\ &= (eba)c \\ &= bac \end{aligned}$$

So N is quasi weak commutative near – ring.

Proposition:2.8
Any quasi weak commutative near – ring with right identity is weak commutative.

Proof:
Let $a, b, c \in N$ and $e \in N$ be a right identity.

Then

$$\begin{aligned} (abc) &= (abc)e \\ &= a(bce) \\ &= a(cbe) && \text{(quasi weak commutative)} \\ &= (acb)e \\ &= acb \end{aligned}$$

This proves N is weak commutative.

Theorem:2.9
Let N be a regular quasi weak commutative near – ring. Then

1. $A = \sqrt{A}$, for every N sub-group A of N
2. N is reduced
3. N has (*IFP)

Proof:
Since N is regular, for every $a \in N$, there exists $b \in N$ such that

$$a = aba \qquad (1)$$

Now $a = aba \ (ba)a$ (quasi weak commutative)

$$a = ba^2 \qquad (2)$$

1. Let A be a N-subgroup of N and $a \in \sqrt{A}$.

Then $a^k \in A$ for some positive integer k.

Now from (2)

$$\begin{aligned} a = ba^2 &= b(a)a \\ &= b(ba^2)a \end{aligned}$$

$= b^2a^3$
$= b^2a(a^2)$
$= b^2(ba^2)a^2$
$= b^3a^4$

$= b^{(k-1)}a^k \varepsilon$ NA which is a subset of A.

Therefore $\sqrt{A}$ is a subset of A.

Obviously A is a subset of $\sqrt{A}$.

This completes the proof of (i).

1. If $a^2 = 0$,then by (2) $a = ba^2 = b0 = 0$.

This completes the proof of (ii).

2. Let ab= 0.

Then $(ba)^2 = (ba)(ba) = b(ab)a = b(0a) = b.0 = 0$ (Proposition 2.2)

So, by (ii) ba = 0

Now for any n ε N,

$$\begin{aligned}(anb)^2 &= (anb)(anb)\\ &= an(ba)nb\\ &= (an)0(nb)\\ &= 0\end{aligned}$$

Again by (ii) anb = 0

This proves (iii).

Theorem:2.10

Let N be a regular quasi weak commutative near – ring. Then every N subgroup is an ideal

1. $N = Na = Na^2 = aN = aNa$
2. for all aϵN

Proof:

Let a ϵ N.

Since N is regular a = aba for some b ε N.

By Result 1.6 ba is an idempotent element. Let ba = e

Then Ne = Nba = Na (by Result 1.6) (1)

Let S = { n-ne/ n ϵ N}. We Claim that (O:S) = Ne.

Now $(n-ne)e = ne-ne^2 = ne-ne = 0$ for all n ϵ N.

So, (n-ne)Ne = 0 (by (ii) of Theorem 2.9).

This implies Ne $\subseteq$ (O:S). (2)

Now let y ϵ (O:S). Then sy = 0 for all s ϵ S.

Since N is regular, $y = yxy$ for some $x \in N$.

Since $yx - (yx)e \in S$, $(yx - (yx)e)y = 0$ (by (1))

(ie) $yxy - yxey = 0$

$y - y(xey) = 0$

$y - y(xye) = 0$

$y - yxye = 0$

$y - ye = 0$

(ie) $y = ye \in Ne$.

It follows that $(O:S) \subseteq Ne$ (3)

From (1), (2) and (3) we get $(O:S) = Ne = Na$.

By Result 1.10, Na is an ideal of N.

Now if M is any subgroup of N, then $M = \sum_{a \varepsilon M} Na$

Thus M becomes an ideal of N.

(2) Since N is regular, for every $a \in N$, there exists $b \in N$ such that

$a = aba$

$= (ba)a$ (R is quasi weak commutative)

$a = ba^2 \in Na^2$

So, $N \subseteq Na^2$ (4)

Now

$Na \subseteq N \subseteq Na^2 \subseteq (Na)a \subseteq Na \subseteq N$

implies $Na = Na^2 = N$ (5)

Now, we shall prove $Na^2 = aNa$

Let $x \varepsilon Na^2$.

Then $x = na^2$ for some $n \varepsilon N$

$= naa$

$= ana$ (Since R is quasi weak commutative)

So, $x \in aNa$.

(ie) $Na^2 \subseteq aNa$ (6)

Let $y \varepsilon aNa$.

Then $y = ana$ for some $n \in N$

$= (na)a$ (Since R is quasi weak commutative)

$= na^2 \in Na^2$

(ie) $aNa \subseteq Na^2$ (7)

Hence

$aNa = Na^2$ (8)

We claim that $aN = aNa$.

Since Na is an ideal (by (i)), for every $a \in N$, $(Na)N \subseteq Na$ (9)

Also for every $n \in N$,

$an = (aba)n = a(ban) \in a(NaN) \subseteq aNa$ (using (9))

Thus $aN \subseteq aNa$ (10)

Obviously $aNa \subseteq aN$ (11)

Hence $aN = aNa$ (12)

Thus from (5), (8) and (12) we get

$N = Na = Na^2 = aN = aNa$ for all $a \in N$.

Definition:2.11

A near – ring N is said to have property P_4, if $ab \in I \Rightarrow ba \in I$, where I is any ideal of N

Theorem:2.12

Let N be a regular quasi weak commutative near – ring. Then

1. Every ideal of N is completely semi – prime
2. N has property P_4

Proof:

Let I be an ideal of N.

1. Let $a^2 \in I$.

Since N is regular,

$$
\begin{aligned}
a &= aba && \text{for some } b \in N \\
&= baa && \text{(since N is quasi weak commutative)} \\
&= ba^2 \in NI \subset I \\
&\Rightarrow a^2 \in I \Rightarrow && a \in I
\end{aligned}
$$

So every ideal of N is completely prime.

2. Let $ab \in I$.

Then $(ba)^2 = (ba)(ba) = b(ab)a \in N\,I\,N \subseteq I$ (by result 1.4)

Then by (i) $ba \in I$.

Thus $ab \in I \Rightarrow ba \in I$.

Thus N has prtoperty P_4.

Theorem:2.13

Let N be a quasi weak commutative near – ring.For every ideal I of N, (I:S) is an ideal of N where S is any subset of N.

Proof:

Let I be an ideal of N and S be any subset of N.

By Result 1.5 (ii), (I:S) = { $n \in N$ / $nS \subseteq I$} is a left ideal of N

Let $s \in S$, and $a \in (I:S)$, then as $\in$ I.

Since N has proprty P_4, sa $\in$ I.

Then for any $n \in N$, $(sa)n \in I$

(ie) $s(na) \in I$

This implies $(an)s \in I$ (N has property P_4)

(ie) an $\in$ (I:S) for any $n \in N$ and hence (I:S) is a right ideal.

Consequently (I:S) is an ideal.

This completes the proof.

Proposition:2.14

Let N be a quasi weak commutative near – ring. For any ideal I of N, and $x_1,x_2,x_3,\ldots..x_n \in N$, If $x_1 .x_2 .x_3 \ldots..x_n \in I$, then $<x_1> <x_2> <x_3>\ldots\ldots<x_n> \subseteq I$.

Proof:

Let $x_1 .x_2 .x_3 \ldots..x_n \in I$.

$\Rightarrow x_1 \in (I : x_2 .x_3 \ldots.x_n)$

$\Rightarrow <x_1> \subseteq (I : x_2 .x_3 \ldots.x_n)$

$\Rightarrow <x_1> x_2 .x_3 \ldots.x_n \subseteq I$ (By proposition2.12)

$\Rightarrow x_2 \in (I : x_3 \ldots.x_n <x_1>)$

$\Rightarrow <x_2> \subseteq (I : x_3 \ldots.x_n <x_1>)$

$\Rightarrow <x_2>x_3x_4 \ldots.x_n<x_1> \subseteq I$

$\Rightarrow x_3x_4 \ldots.x_n<x_1> <x_2> \subseteq I$

Continuing like this we get $<x_1> <x_2>\ldots..<x_n> \subseteq I$

Proposition:2.15

Let N be a quasi weak commutative near – ring. Then

1. N has strong IFP
2. N is semi – prime near - ring

Proof:

1. Let I be an ideal of N. Since N is zero symmetric, $NI \subseteq I$.

By Theorem 2.10 (ii), $aN = Na^2$.

Hence $an = ma^2$ for some $m, n \in N$.

Hence, if $ab \in I$, then for every $n \in N$, $anb = ma^2b$

$= (ma)ab \in NI \subset I$

(ie) $ab \in I \Rightarrow anb \in I$ for all $n \in N$.

This proves N has strong IFP.

2. Let M be an N subgroup of N. Then M is an ideal by Theorem 2.10 (i)

Let I be any ideal of N such that $I^2 \subseteq M$.

Then by Result 1.4 NI $\subseteq$ I.
If $a \in I$, then $a = aba$ for some $b \in N$ (N is regular)
$\Rightarrow$ $a = aba \in I(NI) \subseteq I^2 \subseteq M$.

So, any N- subgroup M of N is a semi – prime ideal. In particular {0} is a semi – prime ideal and hence N is a semi – prime near – ring

References

[1] Dheena P. On strongly regular near – rings, Journal of the Indian Math.Soc., 49(1985), 201 – 208.
[2] Dheena P.A note on a paper of Lee, Journal of Indian Math.Soc., 53(1988), 227 – 229.
[3] Fain, Some Structure theorems for near – rings, Doctoral dissertation, University of Oklanama,1968.
[4] Gratzer.George, Universal Algebra, Van Nostrand, 1968.
[5] Henry E. Heartherly, Regular Near – rings, Journal of Indian Maths.Soc,38(1974),345-354.
[6] Oswald A.Near – rings in which every N – subgroup is principal, Proc . London Math.Soc, 3 (1974), No 28, 67 – 88.
[7] Pliz . Giinter, Near – rings, North Holland, Aneterdam, 1983.
[8] Uma.S, Balakrishnan.R and Tamizh chelvam.T, Pseudo Commutative near – rings, Scientia Magna, 6(2010) No 2, 75 – 85.