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Expected Number of Real Zeros of Random Ultraspherical Polynomial

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Abstract

Let y_0 (w) , y_1 (w) , …, y_n (w) be a sequence of independent random variables with mathematical expectation zero and variance one. We show that the polynomial $\sum_{k=0}^{n} y_k$ (w) ψ_k (t) , where ψ_k (t) is the normalized orthogonal ultraspherical polynomial, has $n/\sqrt{3}$ real zeros for large values of n.

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1.Introduction

Let f_n (t) = $\sum_{k=0}^{n} b_k y_k$ (w) ϕ_k (t) be a random polynomial, where y_0 (w) , y_1 (w) , … y_n (w) is a sequence of mutually independent, normally distributed random variables with mathematical expectation zero and variance one ; (ϕ_0 (t) , ϕ_1 (t) , ...) is a sequence of real valued polynomials (functions) and (b_0, b_1, …) a sequence of real constants. Kac [4] showed that, when b_k=1 and ϕ_k (t) =t^k, the expected number of times the random polynomial f_n (t) crosses the t- axis is asymptotic to $(1/\pi)\log n$. When b_0=0, b_k=1 for k≠0 and ϕ_k (t) = cosk $(\cos^{-1} t)$, J.E.A. Dunnage [2] estimated that the expected number of crossings of f_n (t) is asymptotic to 2n/√3 in the interval (-1, 1) .

It is interesting to observe that while $t^{k,}$ s are a set of functions monotonic in (-∞, 0]and [0, ∞) , cosk $(\cos^{-1} t)$ for each k, oscillates k times between -1 and 1. The fact that the average number of zeros of y=0 when ϕ_k (t) = cosk $(\cos^{-1} t)$ is proportional to the number of individual oscillations of ϕ_k (t) about the t-axis, draws attention to the question as to how far the oscillatory nature of $\emptyset_k(t)$ decisively affects the zeros of y=o. Although the answer remains still inconclusive, we attempt to show that for large n, the above equation may be expected to have c.n, (c>0) number of real roots when ϕ_k (t) happens to be the ultraspherical classical orthogonal Gegebauer polynomial. In other words the oscillatory property of ϕ_n (t) is also shared by $\sum_{k=0}^{n} y_k$ (w) $b_k\phi_k$ (t) . It was Das[1] who first initiated the work on Random Orthogonal polynomial. Taking $b_k\phi_k$ (t) as normalized orthogonal Legendre Polynomial, he showed that expected number of real zeros of the polynomial is asymptotic to $n/\sqrt{3})$. Das's polynomial can be considered as a special case of our polynomial $f_n(t)$.

Now, ϕ_k (t) is associated with a weight function u (t) = $(1-t^2)^{-1/2}$, t>1/2 corresponding to the interval (-1, 1) over which the integral of u (t) ϕ_k^2 (t) is a positive number h_k .We take $b_k=h_k^{1/2}$.Then the integral of ψ_k^2 (t) $=b_k^2\phi_k^2$ (t) over the given interval is unity, so that each of the terms of the polynomial $\sum_{k=0}^{n} y_k$ (w) ψ_k (t) $= \sum_{k=0}^{n} y_k$ (w) $b_k\phi_k$ (t) has same weightage in the same sense.

Thus, in what follows, we find the average number of zeros of the equation

$$f_n(t) = \sum_{k=0}^{n} y_k (w)\, \psi_k (t) =0. \tag{1.1}$$

Let EN_n (α, β) denote the expected number of real zeros of f_n (t) in (α, β) . We prove the following theorem

THEOREM: If $\{y_k (w)\}^n_{k=0}$ is a sequence of mutually independent, normally distributed random variables with mean zero and variance unity;. Then

$$EN_n(-\infty,\infty) \sim \frac{n}{\sqrt{3}}.$$

2. FORMULA FOR EN_n (α, β)

Following the procedure of Kac [4], we obtain

$$EN_n (f;a, b) = \frac{1}{\pi}\int_a^b \frac{[\{X_n (t)\}\{Z_n (t)\}-\{Y_n (t)\}^2]^{1/2}}{X_n(t)}\,dt \tag{2.1}$$

where

$X_n (t) = X_n = \sum_{k=0}^{n} [\psi_k (t)]^2$

$Y_n (t) = Y_n = \sum_{k=0}^{n} [\psi_k (t)][\psi'_k (t)]$

$Z_n (t) = Z_n = \sum_{k=0}^{n} [\psi'_k (t)]^2$

provided that $X_nZ_n-Y_n>0$ which holds good by Cauchy's inequality.

Let us put $\mu_k=r_nh_n^{-1}r_{n+1}^{-1}$ where r_n is the coefficient of t^n in ϕ_n (t) .The famous Christofell-Darboux formula ([3], p.159) of the theory of orthogonal functions reads as follows :

$$\sum_{k=0}^{n} h_k^{-1}\phi_k(\mu)\,\phi_k(t) = \mu_n\,\frac{\{\phi_{n+1}(\mu)\phi_n(t)-\phi_n(\mu)\phi_{n+1}(t)\}}{\mu-t} \tag{2.2}$$

Putting $\mu=t+\gamma$, expanding both the sides of (3.1.3) by Taylor's series and equating the coefficients of γ we obtain that

$$\sum_{k=0}^{n}[\psi_k(t)]^2=\sum_{k=0}^{n}h_k^{-1}[\phi_k(t)]^2=\mu_n[\phi_{n+1}'(t)\,\phi_n(t)-\phi_{n+1}(t)\,\phi_n'(t)], \quad (2.3)$$

$$\sum_{k=0}^{n}[\psi_k(t)][\psi_k'(t)]=\sum_{k=0}^{n}h_k^{-1}[\emptyset_k(t)\,\emptyset_k'(t)]=\frac{\mu_n}{2}[\emptyset_{n+1}''((t)\,\emptyset_n(t)-\phi_{n+1}(t)\,\emptyset_n''(t)], \quad (2.4)$$

$$\sum_{k=0}^{n}h_k^{-1}[\phi_k(t)\,\emptyset_k''(t)]=\frac{\mu_n}{3}[\emptyset_{n+1}''(t)\,\phi_n(t)-\phi_{n+1}(t)\,\emptyset_n'''t)]$$

Differentiating (2.4), we get

$$\sum_{k=0}^{n}h_k^{-1}[\emptyset_k'^{\,2}(t)+\emptyset_k''(t)\,\emptyset_k(t)]=\frac{\mu_n}{2}[\emptyset_{n+1}'''(t)\,\emptyset_n(t)+\emptyset_n'(t)\,\emptyset_{n+1}''(t)-\emptyset_{n+1}'(t)\,\emptyset_n''(t)-\emptyset_n'''(t)\,\emptyset_{n+1}(t)].$$

Therefore,

$$\sum_{k=0}^{n}h_k^{-1}[\emptyset_k'^{\,2}(t)=\sum_{k=0}^{n}[\psi_k'(t)]^2=\frac{\mu_n}{2}[\emptyset_{n+1}''(t)\,\emptyset_n'(t)-\emptyset_{n+1}'(t)\,\emptyset_n''(t)]+\frac{\mu_n}{6}[\emptyset_{n+1}'''(t)\,\phi_n(t)-\phi_{n+1}(t)\,\emptyset_n'''(t)]. \quad (2.5)$$

Thus

$$EN_n(f, a, b)=\frac{1}{\pi}\int_a^b g_n(t)\,dt, \quad (2.6)$$

where

$$g_n^2(t)=\left[\frac{Wn(t)+V_n(t)}{R_n}-\frac{U_n^2(t)}{4R_n^2(t)}\right],$$

$R_n(t)=\emptyset_{n+1}'(t)\,\emptyset_n(t)-\emptyset_{n+1}(t)\,\emptyset_n'(t)$, $U_n(t)=\emptyset_{n+1}''(t)\,\emptyset_n(t)-\emptyset_{n+1}(t)\,\emptyset_n''(t)$, $V_n(t)=1/2[\emptyset_{n+1}''(t)\,\emptyset'_n(t)-\emptyset_{n+1}'(t)\,\emptyset_n''(t)]$ and $W_n(t)=1/6[\emptyset_{n+1}'''(t)\,\emptyset'_n(t)-\emptyset_{n+1}(t)\,\emptyset_n''(t)]$.

3.Proof of the theorem

For the sake of convenience for proving the theorem, we break up the interval (-1, 1) into three sub intervals namely (i) $(-1+\epsilon, 1+\epsilon)$, (ii) $(-1, -1+\epsilon)$, and (iii) $(1-\epsilon, 1)$. We choose $\epsilon=n^{-1/(4+\delta)}$.

In the section 3.1, we find out the average number of zeros in the first interval. In the section 3.2 We prove that the number of zeros in the other two intervals are negligible in comparison to those in the interval (i).

EXPECTED NUMBER OF ZEROS IN THE INTERVAL $(-1+\epsilon, 1-\epsilon)$

From [17,], We have

$$(1-t^2)\,\emptyset_n''(t)=(2\lambda+1)\,t\emptyset_n'(t)-n\,(n+2\lambda)\,\phi_n(t) \quad (3.1)$$

Thus

$$2\,(1-t^2)\,V_n(t)=n\,(n+2\lambda)\,R_n(t)-(2n+1+2\lambda)\,\phi_{n+1}(t)\,\emptyset_n'(t), \quad (3.2)$$

$$(1-t^2)\,U_n(t)=(2\lambda+1)\,t\,R_n(t)-(2n+1+2\lambda)\,\phi_{n+1}(t)\,\phi_n(t). \quad (3.3)$$

Differentiating (3.1) we have

$$-2t\emptyset_n''(t) + (1-t^2)\,\emptyset_n'''(t) = (2\lambda+1)\,\emptyset_n'(t) + (2\lambda+1)\,t\emptyset_n''(t) - n(n+2\lambda)\,\emptyset_n'(t).$$

Therefore it is easy to derive that

$$6(1-t^2)\,W_n(t) = (2\lambda+3)t\left[\frac{(2\lambda+1)t}{(1-t^2)}R_n(t) - \frac{(2n+1+2\lambda)}{(1-t^2)}\emptyset_{n+1}'(t)\emptyset_n(t)\right]$$
$$+[(2\lambda+1) - n(n+2\lambda)]R_n(t)$$
$$+\frac{(2n+1+2\lambda)}{(1-t^2)}\emptyset_n(t)\left[(n+1)t\emptyset_{n+1}(t) - \big(-(2\lambda+1)\big)\emptyset_n(t)\right]. \qquad (3.4)$$

From [17, p . 279], we have

$$(1-t^2)\,\emptyset_{n+1}'(t)\,) = (2\lambda+n)\emptyset_n(t) - t\emptyset_{n+1}(t)$$

Therefore

$$(1-t^2)R_n(t)$$
$$= (\lambda+n)\emptyset_n^2(t) - t\emptyset_{n+1}(t)\emptyset_n(t) - (2\lambda+1+n)\emptyset_{n-1}(t)\emptyset_{n-1}(t). \qquad (3.5)$$

Gegenbauer polynomials can be considered as a special case of the Jacobi Polynomial $P_n^{(\alpha,\beta)}$ (t) , where $\alpha = \beta = \lambda - \frac{1}{2}$ (see Szego[5] page 80].For large n, We shall use the asymptotic estimate of ϕ_n (t) as

$$\phi_n(t) \sim \frac{2^{\lambda}}{(\pi n)^{1/2}}\,(1-t)^{\frac{-\lambda}{2}}\,(1+t)^{\frac{-\lambda}{2}}\left[\cos\chi + o\left(\frac{1}{n\sin\theta}\right)\right], \qquad (3.6)$$

where $\chi = (n\theta+\lambda\theta-\lambda\pi/2)$ and $t=\cos\theta$. (We have taken $\alpha = \beta = \lambda - \frac{1}{2}$ in asymptotic estimate of $P_n^{(\alpha,\beta)}$ (Szego[5] page 190)

Hence from (3.5) and (3.6) we have

$$R_n(t) = \frac{2^{2\lambda}}{\pi(1-t^2)}\,(1-t)^{-\lambda}\,(1+t)^{-\lambda}\left[(1-t^2) + o\left[\frac{1}{n\sin\theta}\right]\right]$$

It is easy derive from (3.6) that

$$\frac{\emptyset_{n+1}(t)\emptyset_n(t)}{R_n(t)} = o\left(\frac{1}{n}\right),\ \frac{\emptyset_n^2(t)}{R_n(t)} = o\left(\frac{1}{n}\right) \text{ and } \frac{\emptyset_n'(t)\emptyset_{n+1}(t)}{R_n(t)} = 0\left(\frac{1}{(1-t^2)}\right).$$

Therefore, from (3.2) and (3.4) we have

$$\frac{V_n(t)}{R_n(t)} = \frac{n(n+2\lambda)}{2(1-t^2)} - \frac{(2n+1+2\lambda)}{2(1-t^2)}\,\frac{\emptyset_{n+1}(t)\emptyset_n'(t)}{R_n(t)} = \frac{n^2}{2(1-t^2)} + 0\left(\frac{n}{(1-t^2)^2}\right),$$

$$\frac{w_n(t)}{R_n(t)} = \frac{[\{\frac{(2\lambda+3)(2\lambda+1)t^2}{1-t^2} + (2\lambda+1) - n(n+2\lambda)\}R_n(t)\,) - \left\{\frac{(2n+1+2\lambda)(2\lambda+3)}{1-t^2} - \frac{(n+1)(2n+1+2\lambda)t}{1-t^2}\right\}\emptyset_{n+1}(t)\emptyset_n(t) - \{\frac{(2n+1+2\lambda)(2\lambda+1)\emptyset_n^2(t)}{1-t^2}\}]}{[6(1-t^2)R_n(t)]} = -$$
$$\frac{n^2}{6(1-t^2)} + 0\left(\frac{n}{(1-t^2)^2}\right).$$

Thus,

$$\frac{W_n(t)+V_n(t)}{R_n} = \frac{n^2}{2(1-t^2)} + 0\left(\frac{n}{(1-t^2)^2}\right).$$

Also, we can derive from (3.3) that

$\frac{U_n(t)}{2R_n(t)}$=0 $(\frac{1}{(1-t^2)^{2}})$ +0$\left(\frac{1}{n(1-t^2)^{2}}\right)$.

So

$$g_n(t) = \sqrt{\frac{W_n(t)+V_n(t)}{R_n(t)} - \frac{U_n^2(t)}{4R_n^2(t)}}$$

$$=\frac{n}{\sqrt{3}(1-t^2)^{1/2}}\left[1+0\left(\frac{1}{n(1-t^2)}\right)\right]^{1/2}$$

For the range (-1+∈, 1-∈) , we notice that 1-$t^2 > 2 \in -\in^2$=$\frac{2n^{(4+\hat{o})^{-1}}-1}{n^{2(4+\hat{o})^{-1}}}$, where ∈= $n^{(4+\hat{o})^{-1}}$, as previously specified . Thus $(1-t^2)^{-1}$=$0\left(n^{(4+\hat{o})^{-1}}\right)$.

Therefore,

$$g_n(t) = \frac{n}{\sqrt{3(1-t^2)^{\frac{1}{2}}}}\left[1+0\left(n^{-\frac{2+\delta}{4+\delta}}\right)\right].$$

Thus from (2.6)

$EN_n(f; -1+\in, 1-\in)$

$$=\int_{-1+\in}^{1-\in}\frac{n}{\pi\sqrt{3}(1-t^2)^{\frac{1}{2}}}dt\left[1+0\left(n^{-\frac{2+\delta}{4+\delta}}\right)\right]=\frac{n}{\sqrt{3}}\left[1+0\left(n^{-\frac{2+\delta}{4+\delta}}\right)\right]$$

as $\sin^{-1}(1-\in)) \sim \pi/2$.

Number of zeros in (-1, -1+€) , and (1-€, 1) .

Here we show that in the ranges (1-∈, 1) and (-1, -1+∈) the number of zeros of (1 . 1) is negligibly small in comparison to those in $(-1+€, 1-€)$.

Let

f (z) =f $(\bar{y}(w), z) = \sum_{k=0}^{n} y_k(w)\psi_k(z)$, (3.7)

where $\bar{y}(w)$ denotes the random vector $(y_0(w), y_1(w), \ldots\ldots\ldots\ldots, y_n(w))$ .Now f $(\bar{y}(w), 1) = \sum_{k=0}^{n} y_k(w)\psi_k(1)$ is a random variable with mean zero and variance $\sigma^2 = \sum_{k=0}^{n}\psi_k^2(1) \geq \psi_0^2(1) \geq 0$and hence has the distribution function $\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{t} exp\left(\frac{-v^2}{2\sigma^2}\right)$dv.

Now

p (|f (1) |≤ $e^{-2n\in}$) $= \left(\frac{2}{\pi\sigma^2}\right)^{1/2}\int_0^{e^{-2n\epsilon}} exp\left(\frac{-v^2}{2\sigma^2}\right)$dv $\leq \left(\frac{2}{\pi\sigma^2}\right)^{1/2} e^{-2n\epsilon} < e^{-n\in}$.

(3.8)

Let $I_n = \max_{0\leq k\leq n}(|y_k(w)|)$. Then

p$(I_n \leq n)$=p$(\prod_{k=0}^{n}|y_k(w)| \leq n)$=$\prod_{k=0}^{n}(p|y_k(w)| \leq n)$ =$(\prod_{k=0}^{n}(1-p(|y_k(w)| > n))$

$$=\left[1-\sqrt{\frac{2}{\pi}}\int_n^\infty e^{-v^2/2}dv\right]^{n+1} > 1-e^{-n^2/2}\ (n > n_0). \quad (3.9)$$

Let $T_n = \max_{0\le k\le n} |\psi_k(1+2\epsilon e^{i\theta})|$.For the Gegenbauer polynomials, $h_n = \frac{2^{1-2\lambda}\pi\Gamma\,(n+2\lambda)}{(n+\lambda)\{\Gamma(\lambda)\}^{\frac{1}{2}}\Gamma\,(n+1)}$ for $\lambda>1/2$[see 17, p.281].Hence $b_n = h_n^{-1/2} < \alpha_1 n^{1/2}$ where α_1 is a constant.From the integral representation of Gegenbauer polynomial [3], we have $\emptyset_n(t) = \frac{2^{1-2\lambda}\Gamma\,(2\lambda+n)}{n!\{\Gamma(\lambda)\}^2}\int_0^{\pi}\,(t+i\sqrt{(1-t^2)}cos\theta)^{\,n}\,(sin\theta)^{\,2\lambda-1}d\theta.$

Remembering that $\in = n^{-1/\,(4+\delta)}$, we have from above representation

$\left|\emptyset^n\,(1+2\in e^{i\theta})\,\right| < \frac{2^{1-2\lambda}\Gamma\,(2\lambda+n)}{n![\Gamma\,(\lambda)\,]^2}\,(1+2\in)^{\,n} < \alpha_3 n^{\alpha_2}\,(1+2\in)^{\,n}) < \alpha_3 n^{\alpha_2}\exp\left[2n^{\frac{3+\delta}{4+\delta}}\right]$, where α_2 and α_3 are constants involving λ only .

Hence

$$T_n = \emptyset_n(t)h_n^{-1/2} < An^{\alpha_2+1/2}\exp\left(2n^{\frac{3+\delta}{4+\delta}}\right), \tag{3.10}$$

where A is a constant .Also

$\left|f\,(1+2\in e^{i\theta})\,\right| = \left|\sum_{k=0}^{n} y_k(w)\psi_k\,(1+2\in e^{i\theta})\,\right| \le \sum_{k=0}^{n}{}_k\,|y_k\,(w)\,|\left|\psi_k\,(1+2\in e^{i\theta})\,\right| \le 1_n\sum_{k=0}^{n} T_n = nl_nT_n$.Hence from (3 . 9) , it follows that P $(\left|f\,(1+2\in e^{i\theta})\,\right| \le n^2T_n) \ge 1-e^{-\frac{n^2}{2}}$.

This together with (3 . 10) , gives

$$\mathrm{P}\left(f\left|1+2\in e^{i\theta}\right| \le An^{\alpha}\exp\,\left(2n^{\frac{3+\delta}{4+\delta}}\right)\right) \ge 1-e^{-\frac{n^2}{2}}, \tag{3 . 11}$$

where $\alpha = \alpha_2 + 5/2$.

So from (3 . 8) and (3 . 11) , we obtain

$$\mathrm{P}\left(\left|\frac{f\,(1+2\in e^{i\theta})}{f\,(1)}\right| \le \mathrm{A}n^{\alpha}\exp\,\left(2\mathrm{n}^{\frac{3+\delta}{4+\delta}} + 2\mathrm{n}\in\right)\right) \ge > 1-e^{-\frac{n^2}{2}-n^{-n\in}} > 1-\frac{2}{n}. \tag{3 . 12}$$

Let n (∈) denote the number of zeros of f $(\bar{y}(w), z)$ =0inside the circle $|z-1| \le \in$ .It is easy to see that the number of zeros of (2 .1) inside the interval 1-∈$\le t \le 1$ does not exceed n (∈) .

By Jensen's theorem, (Tetchmarsh[6]) we have

$n\,(\in) \le \frac{1}{2\pi log2}\int_0^{2\pi} log\left|\frac{f\,(1+2\in e^{i\theta})}{f\,(1)}\right| d\theta \le \frac{1}{2\pi log2}\int_0^{2\pi} log\left\{An^{\alpha}exp\left(2n^{\frac{3+\delta}{4+\delta}}\right) + 2n\in\right\} d\theta,$

for f (1) $\neq 0$, except for a set of measure at most 2/n, as is evident from (3 .12) .

Thus, we obtain that the number of zeros of (1 .1) in (1-∈, 1) is at most $0\left(n^{\frac{3+\delta}{4+\delta}}\right)$, with probability at least $1-\frac{2}{n}$.Identical result is obtainable for the number of zeros of (1.1) in (-1, -1+∈) , so that

EN_n (f;-1, -1+∈) =0$\left(n^{\frac{3+\delta}{4+\delta}}\right)$.

The above derivation together with the estimate of $EN_n(f; -1+\in, 1-\in)$, proves that

$$EN_n\ (f; -1, 1) = \frac{n}{\sqrt{3}} + 0\left(n^{\frac{3+\delta}{4+\delta}}\right).$$

Reference

[1] Das M., Real zeros of a random sum of orthogonal polynomials, Proc. Amer. Math. Soc. 27 (1971) , 147{153.

[2] Dunnage, J.E.A .The number of Real Zeros of Random Trigonometric Polynomial, Proc. London Math. Soc. (3) 16 (1966)

[3] Erdélyi, A "Higher Transcendental Functions", Vol-II, Mc. Graw Hill Book Company, New York, 1953

[4] Kac M.On the average number of real roots of a random algebraic equation, Bull. Amer. Math. Soc. 49 (1943) , 314-320. MR0007812

[5] Szeg□o G. "Orthogonal Polynomials", Amer. Math. Soc., Rhode Island, 1939.53-54

[6] Titchmarsh, E.C." Theory of functions", 2nd Edition, The English Language Book Society, Oxford University Press