International Journal of Mathematics Research.
ISSN 0976-5840 Volume 5, Number 5 (2013), pp. 461-473

http://www.irphouse.com

Characterization of Wiener Process by Symmetry and Constant Regression

Dr. T. Vasanthi and Mrs. M.Geetha

Associate Professor of Mathematics, ADM College for women, Nagapattinam
Assistant Professor of Mathematics, Saradha Gangadharan College, Pondicherry.

Abstract

This paper establishes the characterization ofWiener process by constant regression.

Keywords: Wiener process,Characteristic function.

1. Introduction Abstract

Several characterization theorems for Wiener Process are now known by means of conditions on either independence or identical distribution of stochastic integrals or through regression properties of them. Some of these results can be found in Lukacs (1968), Ramachandran and Rao (1970) and Prakasa Rao (1970). Our aim in this paper is to obtain another characterization of Wiener process paralleling the following theorem of Hyede (1970). Some preliminaries on stochastic processes are explained in section 2 and section 3 contains the main characterization theorem.

2.Characterization of Wiener Process by Symmetry

Theorem 1.1 (Hyede)

Let $X_1, X_2, \ldots\ldots\ldots\ldots\ldots, X_k$ be independent random variables and $a_j, b_j, 1 \leq j \leq k$ be non – zero constants with $a_i b_i^{-1} = -a_j b_j^{-1}$ for all $i \neq j$. If the conditional distribution of $\sum_{j=1}^{k} a_j X_j$ given $\sum_{j=1}^{k} b_j X_j$ is symmetric , then each X_j has a normal (or degenerate) distribution. these distributions are constrained in such a way that if $E\{\exp{(it)X_j}\} = exp(it\, A_j - B_j t^2)$, $1 \leq j \leq k, A_j, B_j$ real, $B_j > 0$, then

$$\sum_{j=1}^{k} A_j a_j = 0 \text{ and } \sum_{j=1}^{k} B_j a_j b_j = 0.$$

Some preliminaries on stochastic processes are explained in section 2 and section 3 contains the main characterization theorem.

Let $T = [A, B]$. Let us consider Stochastic Processes $\{X(t), t \in T\}$. Let $a(\cdot)$ be a continuous function on $[A, B]$. It can be shown that the integral

$$\int_A^B a(t) dX(t)$$

exists as the limit in the sense of convergence in probability of the corresponding Riemann – Stieltjes sums if the process is continuous and homogeneous with independent increments as defined below.

A Stochastic Process $\{X(t), t \in T\}$ is said to be homogeneous process with independent increments if the distribution of the increments $X(t+h) - X(t)$ depends on h but not t and if the increments over non – overlapping intervals are stochastically independent. the process is said to be continuous if $X(t)$converges in probability to $X(s)$ as t tends to s for every $s \in T$. let $\phi(u, h)$ denote the characteristic function of $X(t+h) - X(t)$. it is well known that $\phi(u; h)$ is infinitely divisible and $\phi(u; h) = \{\phi(u; 1)\}^h$. a homogeneous process $\{X(t), t \in T\}$ with independent increments is called a Wiener process if the increments $X(t+h) - X(t)$ are normally distributed with variance proportional to h.

Lemma 1.1

Let $\{X(t), t \in T\}$ be a continuous homogeneous process with independent increments on $T = [A, B]$. Let $a(\cdot)$ and $b(\cdot)$ be continuous functions on $[A, B]$. Let $Y = \int_A^B Q(t) dX(t)$ and $Z = \int_A^B b(t) dX(t)$. Let $\phi(u; h)$ and $\theta(u, v)$ be the characteristic functions of $X(t+h) - X(t)$ and (Y, Z) respectively. Then $\theta(u, v)$ is different from zero for all u and v and

$$\log\, \theta(u, v) = \int_A^B \psi[u\, a(t) + v b(t)]\, dt \tag{1.1}$$

where $\psi(u) = \log \phi(u; 1)$.

Lemma 1.2

If $f(y)$ is absolutely integrable with respect to a measure P and $\int f(y) e^{iuy} P(dy) = 0$ for all u, then $f(y) = 0$ $a.e.$ P.

Theorem 1.2

let $\{X(t), t \in T\}, T = [A, B]$ be a continuous homogeneous process with independent increments and suppose that the increments have non – degenerate distributions. let $a(\cdot)$ and b$(\cdot)$ be continuous functions defined on $[A, B]$ with the property that

$$\int_A^B |b^3(t)[a(t)]^{-1}|dt < \infty \; and \int_A^B b^3(t)[a(t)]^{-1}dt \neq 0 \tag{1.2}$$

Let

$$Y = \int_A^B a(t)dX(t)\,; Z = \int_A^B b(t)dX(t) \tag{1.3}$$

be defined in the sense of convergence in probability. Then the conditional distribution of Y given Z is symmetric if and only if $\{X(t), t \in T\}$ is a Wiener Process with mean function $m(t) = \lambda t$ and $a(\cdot)$ and $b(\cdot)$ satisfy the relations

$$\lambda \int_A^B a(t)dt = 0 \;\; and \;\; \int_A^B a(t)b(t)dt = 0 \tag{1.4}$$

Proof

"If part". Let Y and Z be as defined in (1.3). Let $\psi(\cdot)$ denote the logarithm of the characteristic function $X(t+1) - X(t)$. Suppose that the conditional distribution of Y given Z is symmetric. Hence the characteristic function of the conditional distribution of Y given Z is real i.e.

$$E\left[e^{iuY}|Z\right] = E\left[e^{-iuY}|Z\right] \tag{1.5}$$

for all real u. this in turn implies that

$$E\left[e^{ivZ+iuY}\right] = E\left[e^{ivZ-iuY}\right] \tag{1.6}$$

for all real numbers u and v. therefore, by lemma (1.1) it follows that

$$\int_A^B \psi(u\,a(t) + vb(t))\,dt = \int_A^B \psi(-u\,a(t) + vb(t))\,dt \tag{1.7}$$

for all u and v.

Now, Let us obtain the relation

$$\int_A^B [\psi(u\,a(t) + vb(t)) - \psi(-u\,a(t) + vb(t))]\,dt = 0 \tag{1.8}$$

for all u and v. Multiplying on the left hand side of (1.8) by $(x-u)$ and integrating from 0 to x, we have

$$\int_0^x \left\{(x-u)\int_A^B [\psi(u\,a(t) + vb(t)) - \psi(-u\,a(t) + vb(t))]\,dt\right\}du = 0$$

for all x and v. Changing the order of integration, let us get

$$\int_A^B \left\{(x-u)\int_0^x [\psi(u\,a(t) + vb(t)) - \psi(-u\,a(t) + vb(t))]\,du\right\}dt = 0 \tag{1.9}$$

for all x and v.Let us now make the substitution $w = ua(t) + vb(t)$ and

$z = -ua(t) + vb(t)$. In this way let us obtain

$$\int_A^B \left\{ \int_{vb(t)}^{xa(t)+vb(t)} \left(x - \frac{w - vb(t)}{a(t)} \right) \psi(w) dw \right\} \frac{dt}{a(t)} - \int_A^B \left\{ \int_{-xa(t)+vb(t)}^{vb(t)} \left(x + \frac{z - vb(t)}{a(t)} \right) \psi(z) dz \right\} \frac{dt}{a(t)} = 0 \tag{1.10}$$

it can be seen from this relation, that the left hand side of (1.5) can be differentiated, twice under the integral sign with respect to v.In particular, it follows that ψ is differentiable twice everywhere. Differentiating twice with respect to v under the integral sign, it can be shown that

$$\int_A^B \frac{b^2(t)}{a^2(t)} \{\psi(x\,a(t) + vb(t)) - \psi(-x\,a(t) + vb(t))\}\,dt = 2x \int_A^B \frac{b^2(t)}{a(t)} \psi'(vb(t)) dt \tag{1.11}$$

for all x and v. Again, by arguments similar to those given above Laha, it can be shown that the left hand side of (1.11) can be differentiated with respect to v under the integral sign and let us get

$$\int_A^B \frac{b^3(t)}{a^2(t)} \{\psi'(x\,a(t) + vb(t)) - \psi'(-x\,a(t) + vb(t))\}\,dt = 2x \int_A^B \frac{b^3(t)}{a(t)} \psi''(vb(t)) dt \tag{1.12}$$

for all x and v. Substituting $v = 0$ in (1.12), one obtains the relation

$$\int_A^B \frac{b^3(t)}{a^2(t)} \{\psi'(x\,a(t)) - \psi'(-x\,a(t))\}\,dt = 2x\psi''(0) \int_A^B \frac{b^3(t)}{a(t)} dt$$

for all x. Let $\theta(x) = \psi(x) - \psi(-x)$. Then it is clear that

$$\int_A^B \frac{b^3(t)}{a^2(t)} \theta'(x\,a(t))\,dt = x\theta''(0) \int_A^B \frac{b^3(t)}{a(t)} dt \tag{1.13}$$

for all x.

$$i.e \quad \int_A^B \frac{b^3(t)}{a^2(t)} \theta''(x\,a(t))\,dt = \theta''(0) \int_A^B \frac{b^3(t)}{a(t)} dt \tag{1.14}$$

$$\int_A^B \frac{b^3(t)}{a(t)} \left\{ 1 - \frac{\theta''(x\,a(t))}{\theta''(0)} \right\}\,dt = 0 \tag{1.15}$$

for all x. Here $\theta''(0) \neq 0$ since the distribution of $X(t)$ is non – degenerate. It can be seen, by arguments similar to those given that $\theta''(x)/\theta''(0)$ is the characteristic function of a symmetric distribution function L. Hence

$$\frac{\theta''(x)}{\theta''(0)} = \int_{-\infty}^{\infty} \cos xz\, L(dz) \tag{1.16}$$

This relation, together with (1.15), implies that

$$\int_A^B \frac{b(t)}{a(t)}\, dt = \int_A^B \frac{b^3(t)}{a(t)} \left[\int_{-\infty}^{\infty} \cos xa(t)z\, L(dz)\right] dt.$$

Let $c(t) = |a(t)|$. Clearly

$$\int_A^B \frac{b^3(t)}{a(t)}\, dt = \int_A^B \frac{b^3(t)}{a(t)} \left[\int_{-\infty}^{\infty} \cos(xc(t)z)\, L(dz)\right] dt,$$

$$= \int_{-\infty}^{\infty} \cos xz\, G(dz) \tag{1.17}$$

where

$$G(z) = \int_A^B \frac{b^3(t)}{a(t)} L\left(\frac{z}{c(t)}\right) dt.$$

Repeated integration is justified by Fubini's theorem in view of assumption. Let
$G^*(z) = MG(z)$ (1.18)

Where

$$M = \left\{\int_A^B \frac{b^3(t)}{a(t)}\, dt\right\}^{-1}. \tag{1.19}$$

Then, let us obtain ,

$$\int_{-\infty}^{\infty} \cos xz\, G^*(dz) = 1 \tag{1.20}$$

for all x from (1.16). We shall first prove that the function $G^*(z)$ (i) is symmetric in the sense that $G^*(-z) = 1 - G^*(z-0)$, (ii) is of bounded variation and (iii) is right continuous. Let $z_n \uparrow z$ as $n \to \infty$. Then

$$G^*(-z) = M \int_{-\infty}^{\infty} \frac{b^3(t)}{a(t)} L\left(-\frac{z}{c(t)}\right) dt$$

$$= M \int_{-\infty}^{\infty} \frac{b^3(t)}{a(t)} \left\{1 - L\left(\frac{z}{c(t)} - 0\right)\right\} dt$$

since L is symmetric distribution function.

$$\Longrightarrow \qquad G^*(-z) = M \int_{-\infty}^{\infty} \frac{b^3(t)}{a(t)} \left\{1 - \lim_{n\to\infty} L\left(\frac{z_n}{c(t)}\right)\right\} dt$$

$$= M \lim_{n\to\infty} \int_{-\infty}^{\infty} \frac{b^3(t)}{a(t)} \left\{1 - L\left(\frac{z_n}{c(t)}\right)\right\} dt$$

by the dominated convergence theorem. Hence

$$G^*(-z) = \lim_{n\to\infty}\{1 - G^*(z_n)\}$$
$$= 1 - G^*(z-0)$$

by definition. This proves that G^* is symmetric in the sense that $G^*(-z) = 1 - G^*(z-0)$. Now let $[\alpha,\beta]$ be any bounded interval and

$$\alpha = z_0 < z_1 < \cdots \dots \dots < z_k = \beta$$

be any subdivision of $[\alpha,\beta]$. Then

$$\sum_{i=1}^{k}|G^*(z_i) - G^*(z_{i-1})| = \sum_{i=1}^{k}\left|M\int_{-\infty}^{\infty}\frac{b^3(t)}{a(t)}\left\{L\left(\frac{z_i}{c(t)}\right) - L\left(\frac{z_{i-1}}{c(t)}\right)\right\} dt\right|$$
$$\leq |M|\int_{-\infty}^{\infty}\left|\frac{b^3(t)}{a(t)}\right|\sum_{i=1}^{k}\left|L\left(\frac{z_i}{c(t)}\right) - L\left(\frac{z_{i-1}}{c(t)}\right)\right| dt$$
$$= |M|\int_{-\infty}^{\infty}\left|\frac{b^3(t)}{a(t)}\right|\left\{L\left(\frac{\beta}{c(t)}\right) - L\left(\frac{\alpha}{c(t)}\right)\right\} dt$$

since L is a distribution function, which in turn proves that

$$\sum_{i=1}^{k}|G^*(z_i) - G^*(z_{i-1})| \leq |M|\int_{-\infty}^{\infty}\left|\frac{b^3(t)}{a(t)}\right| dt$$

for any subdivision

$$\alpha = z_0 < z_1 < \cdots \dots \dots < z_k = \beta$$

of $[\alpha,\beta]$. Hence G^* is of bounded variation on $[A,B]$. This also shows that the total variation of G^* on $(-\infty,\infty)$ is finite. Right continuity of G^* follows from the right continuity of L and dominated convergence theorem. In fact G^* is continuous at all points at which L is continuous. (1.20) together with the fact that G^* is symmetric gives

$$\int_{-\infty}^{\infty} e^{ixz}\, G^*(dz) = 1 \tag{1.21}$$

for all x where G^* is a function of bounded variation whose total variation on $(-\infty,\infty)$ finite. Let

$$H^*(z) = 0 \qquad for\ z < 0$$
$$= 1 \qquad for\ z \geq 0$$

Clearly,

$$\int_{-\infty}^{\infty} e^{ixz}\, H^*(dz) = 1 \tag{1.22}$$

for all x. (1.21) and (1.22) together show that for all x,

$$\int_{-\infty}^{\infty} e^{ixz}\, G^*(dz) = \int_{-\infty}^{\infty} e^{ixz}\, H^*(dz)$$

where G and H are functions of bounded variation with total variation finite. Hence by remarks in Wintner it follows that

$$G^*(z) - H^*(z)$$

is a constant c on the set of continuity points $G^* - H^*$. From the definition of G^*, let us obtain $G^*(+\infty) = 1$. Clearly $H^*(+\infty) = 1$. Taking limits through continuity points of $G^* - H^*$, let us obtain that the constant c is zero. Hence

$$G^*(z) = H^*(z)$$

at all continuity points of $G^* - H^*$. Since H is continuous at all points except zero, it follows that

$$\begin{aligned} G^*(z) &= 0 \qquad for\, z < 0 \\ &= 1 \qquad for\, z > 0 \end{aligned}$$

where z is any continuity point of G^*. From the right continuity of $G^*$and the fact that the set of continuity points of $G^*$is dense, let us obtain that

$$\begin{aligned} G^*(z) &= 0 \qquad for\, z < 0 \\ &= 1 \qquad for\, z \geq 0 \end{aligned} \tag{1.23}$$

for every x. (1.22) and (1.17) together show that

$$G^*(0) = |M| \int_{-\infty}^{\infty} \frac{b^3(t)}{a(t)} L(0)\, dt = L(0)$$

therefore $L(0)$ is equal to 1. Hence $L(z) = 1$ for $z \geq 0$. From the asymmetry of L, let us obtain that $L(z) = 0$ for $z < 0$.(1.16) shows that

$$\theta''(x\,) = \theta''(0)$$

For all x. It can be shown that $\theta(t)$ is a quadratic polynomial in t and let us conclude from cramer's theorem that the increments of the process are normally distributed. Since ψ is differentiable twice under the integral sign, let us obtain from (1.7) that

$$\int_A^B a(t)\psi'(u\,a(t) + vb(t))\, dt = -\int_A^B a(t)\psi'(-u\,a(t) + vb(t))\, dt \tag{1.24}$$

and

$$\int_A^B a(t)b(t)\psi''(u\,a(t) + vb(t))\, dt = -\int_A^B a(t)b(t)\psi''(-u\,a(t) + vb(t))\, dt \tag{1.25}$$

Substituting $u = v = 0$, it follows that

$$\psi'(0) \int_A^B a(t) dt = 0 \tag{1.26}$$

and

$$\psi''(0)\int_A^B a(t)b(t)dt = 0 \tag{1.27}$$

Since $t\psi' = iE[X(t)] = i\lambda t$ and $\psi''(0) \neq 0$, let us obtain that the functions $a(\cdot)$ and $b(\cdot)$ satisfy (1.4). This completes the proof of the "If part".

"Only If part".

Suppose $\{X(t), t \in T\}$ is a Wiener process with mean $m(t) = \lambda t$ and covariance function $r(s,t) = \sigma^2 \min(s,t)$ where $-\infty < \lambda < \infty, \sigma^2 > 0$.

Let $a(\cdot)$ and $b(\cdot)$ be continuous functions defined on $[A,B]$ and let Y and Z be defined as in (1.3). Further suppose that

$$\lambda\int_A^B a(t)dt = 0 \ \ and \ \int_A^B a(t)b(t)dt = 0 \tag{1.28}$$

Let $\psi(\cdot)$ be the logarithm of the characteristic function $X(t+1) - X(t)$. Then $\psi(t) = i\lambda t - \frac{1}{2}\sigma^2 t^2$. In view of (1.27), it follows that for all u and v,

$$\int_A^B \psi[u\,a(t) + vb(t)]\,dt = \int_A^B \psi[-u\,a(t) + vb(t)]\,dt$$

which in turn implies that

$$E\left[e^{iuY+ivZ}\right] = E\left[e^{-iuY+ivZ}\right]$$

by Lemma (1.1). Hence

$$E\left[e^{ivz}E\left\{e^{iuY}|Z\right\}\right] = E\left[e^{ivz}E\left\{e^{-iuY}|Z\right\}\right] \tag{1.29}$$

for all u and v. This in turn proves that

$$E\left[e^{iuY}|Z\right] = E\left[e^{-iuY}|Z\right]$$

almost everywhere with respect to the distribution of Z by lemma 1.2. Hence the conditional distribution of Y given in Z is symmetric. This completes the proof of the "Only if part".

2. Characterization of Wiener process by constant Regression

Let $T = [A,B]$. Let us consider Stochastic Processes $\{X(t), t \in T\}$ which have finite moments of all orders. In particular, $\{X(t), t \in T\}$ will be a stochastic process of the second order. Let $a(\cdot)$ be a function which is continuous on $[A,B]$, and suppose that the mean function $m(t) = E[X(t)]$ and the covariance function $r(s,t) = E[X(t)X(s)] - E[X(t)]E[X(s)]$ are of bounded variation in $[A,B]$. It can be shown that the integral

$$\int_A^B a(t)\,dX(t) \tag{2.1}$$

exists as the limit in the mean (lim) of the corresponding Riemann – Stieltjes sums.

Definition 2.1
A stochastic process $\{X(t), t \in T\}$ Is said to be a homogeneous process with independent increments if the distribution of the increments $X(t+h) - X(t)$ depends only on h but is independent of t, and if the increments over non – overlapping intervals are stochastically independent. The process is said to be continuous if $X(t)$ converges in probability to $X(s)$ as t tends to s for every $s \in T$. Let $\{X(t), t \in T\}$ be a continuous homogeneous process with independent increments. Let $\varphi(u; h)$ denote the characteristic function of $X(t+h) - X(t)$.

Definition 2.2
A homogeneous process $\{X(t), t \in T\}$ with independent increments is called a wiener process if the increments $X(t+h) - X(t)$ are normally distributed with variance proportional to h.

Lemma 2.1
A random variable y, with finite expectation, has constant regression on a random variable Z, i.e., $E[Y|Z] = E[Y]$ a.e. if and only if

$$E\left[Ye^{itZ}\right] = E[Y]E\left[e^{itZ}\right] \tag{2.2}$$

Lemma 2.2
Let $\{X(t), t \in T\}$be a continuous homogeneous stochastic process with independent increments on $T = [A, B]$. Further, suppose that the process is a second order process and its mean function and covariance function are of bounded variation in $[A, B]$. Let

$$Y = \int_A^B g(t)dX(t)\,; Z = \int_A^B h(t)dX(t)$$

for continuous functions $g(\cdot)$ and $h(\cdot)$ on $[A, B]$. Denote by $\varphi(u; h)$ and $\theta(u, v)$ the characteristic functions of $X(t+h) - X(t)$ and (Y, Z)respectively. Then $\theta(u, v)$ is different from zero for all u and v and

$$\log \quad \theta(u, v) = \int_A^B \psi[ug(t) + vh(t)]\, dt$$

where $\psi(u) = \log \varphi(u; 1)$.

Lemma 2.3
Let $\{X(t), t \in T\}$ be a continuous homogeneous stochastic process with independent increments on $T = [A, B]$. Further, suppose that the process is a second order process and its mean function and covariance function are of bounded variation on $[A, B]$. Let $g(\cdot)$ and $h(\cdot)$ be continuous functions on $[A, B]$. Denote

$$Y = \int_A^B g(t)dX(t)\,; \qquad Z = \int_A^B h(t)dX(t)$$

then for any real number v,

$$E\left[Ye^{ivZ}\right] = -i\left(\int_A^B g(t)\psi'[vh(t)]\,dt\right)\cdot exp\left(\int_A^B \psi[vh(t)]\,dt\right) \tag{2.3}$$

where $\psi(u) = \log\varphi(u;1)$ is the logarithm of the characteristic function $X(t+1) - X(t)$.

Proof

let $\theta(u,v)$ denote the characteristic function of the bivariate random variable (Y,Z). By Lemma 2.2, log $\theta(u,v)$is well – defined and

log $\theta(u,v) = \int_A^B \psi[ug(t)+vh(t)]\,dt,$

i.e., $E[exp(iuY+ivZ)] = exp\left(\int_A^B \psi[ug(t)+vh(t)]\,dt\right)$.

Differentiating on both sides with respect to u, let us get that for all u and v,

$$E[iY\,exp(iuY+ivZ)]$$
$$= exp\left(\int_A^B \psi[ug(t)+vh(t)]\,dt\right)\cdot\int_A^B \psi'[ug(t)+vh(t)]\,g(t)dt \tag{2.4}$$

where $\psi'(u)$ denotes the derivative of $\psi(\cdot)$ at u. This differentiation is valid since the random vector (Y,Z) has moments of all orders. Take $u=0$ in (2.4). then it follows that

$$E\left[iY\,e^{ivZ}\right] = exp\left(\int_A^B \psi[vh(t)]\,dt\right)\cdot\int_A^B \psi'[vh(t)]\,g(t)dt$$

and hence

$$E\left[Ye^{ivZ}\right] = -i\left(\int_A^B g(t)\psi'[vh(t)]\,dt\right)\cdot exp\left(\int_A^B \psi[vh(t)]\,dt\right)$$

which completes the proof of the lemma.

Theorem 2.1

Let $\{X(t), t\in T\}$be a continuous homogeneous stochastic process with independent increments and suppose that the increments have non – degenerate distributions. Further suppose that the process has moments of all orders and its mean function well as its covariance function are of bounded variation in $T=[A,B]$. Let $a(\cdot)$ and $b(\cdot)$ be continuous functions defined on $[A,B]$ with the property that

$$\int_A^B a(t)b(t)dt = 0 \tag{2.5}$$

implies that

$$\int_A^B a(t)[b(t)]^k dt \neq 0 \tag{2.6}$$

for all $k > 1$. Let

$$U = \int_A^B a(t)dX(t) = 0 \tag{2.7}$$

$$V = \int_A^B b(t)dX(t) = 0 \tag{2.8}$$

Then U has constant regression on V, i.e.,

$$E[U|V] = E[U] \quad a.e$$

if and only if

$\{X(t), t \in T\}$ is a Wiener process with a linear mean function,

$$\int_A^B a(t)b(t)dt = 0$$

Proof

"Only If part". Suppose $\{X(t), t \in T\}$ is a Wiener process with mean $m(t) = \lambda t$ and covariance function $r(s,t) = \sigma^2 \min(s,t)$ where $-\infty < \lambda < \infty, \sigma^2 > 0$.Let $a(\cdot)$ and $b(\cdot)$ be continuous functions on $[A, B]$ and define U and V as in the Theorem. Further suppose that $a(\cdot)$ and $b(\cdot)$ are such that

$$\int_A^B a(t)b(t)dt = 0 \tag{2.9}$$

Since $\psi(\cdot)$ is the logarithm of the characteristic function $X(t+1) - X(t)$. It is well known that $\psi(t) = i\lambda t - \frac{1}{2}\sigma^2 t^2$. In order to show that $E[U|V] = E[U]$ a.e. it is enough to prove that,

$$E\left[Ue^{isV}\right] = E[U]E\left[e^{isV}\right] \tag{2.10}$$

by lemma (2.1). By Lemmas (2.3) and (2.2) and the condition (2.9), it follows that

$$\begin{aligned} E\left[Ue^{isV}\right] &= -i \left(\int_A^B a(t)\psi'[sb(t)]\, dt\right) exp\left(\int_A^B \psi[sb(t)]\, dt\right) \\ &= -i \left(\int_A^B a(t)\psi'[sb(t)]\, dt\right) E\left[e^{isV}\right] \\ &= -i \left(\int_A^B a(t)\{i\lambda - \sigma^2 sb(t)\}\, dt\right) E\left[e^{isV}\right] \qquad (2.11) \\ &= -i \left(\int_A^B i\lambda a(t)\, dt\right) E\left[e^{isV}\right] \\ &= \left\{\int_A^B \lambda a(t)\, dt\right\} E\left[e^{isV}\right] \end{aligned}$$

It can be shown easily that $E[U] = \lambda \int_A^B a(t)dt$ which proves that

$$E\left[Ue^{isV}\right] = E[U]E\left[e^{isV}\right]$$

in view of (2.13). This completes the proof of the “only if” part.

“If” part. Let U and V be as defined in the theorem. Let $\psi(.)$ denote the logarithm of the characteristic function $X(t+1) - X(t)$. Further suppose that U has constant regression on V, i.e.,

$$E[U|V] = E[U] \text{ a.e} \tag{2.12}$$

This implies that

$$E\left[Ue^{isV}\right] = E[U]E\left[e^{isV}\right]$$

By lemma 2.1. Hence by Lemma 2.1 and 2.3 it follows that

$$-i\left(\int_A^B a(t)\psi'[sb(t)]\,dt\right) exp\left(\int_A^B \psi[sb(t)]\,dt\right)$$

$$= E\left[Ue^{isV}\right] \tag{2.13}$$

$$= E[U]E\left[e^{isV}\right]$$

$$= E[U] exp\left(\int_A^B \psi[sb(t)]\,dt\right)$$

The above equality gives the relation

$$\int_A^B a(t)\psi'[sb(t)]\,dt = iE[U] \tag{2.14}$$

For any real number s. Since the process has moments of all orders by assumption, $\psi(.)$ has derivatives of all orders and the differentiations with respect to s under integral sign in (2.14) are valid. Differentiating once with respect to s, we get that

$$\int_A^B \psi''[sb(t)]a(t)b(t)\,dt = 0 \tag{2.15}$$

Let s = 0. Then we have

$$\psi''(0)\int_A^B a(t)b(t)\,dt = 0$$

$\psi''(0)$ is different from zero, since by assumption the increments of the process have non – degenerate distributions hence it follows from the above equality, that

$$\int_A^B a(t)b(t)\,dt = 0 \tag{2.16}$$

Differentiating k times with respect to s under the integral sign in (2.15), we obtain that

$$\int_A^B \psi^{(k)}[sb(t)]\,a(t)[b(t)]^{k-1}dt = 0 \tag{2.17}$$

For $k \geq 3$ where $\psi^{(k)}(s)$ denotes the kth derivative of $\psi(.)$ at s. Take s = 0 in (2.17). Then we have,

$$\psi^{(k)}(0) \int_A^B a(t)[b(t)]^{k-1}\, dt = 0 \tag{2.18}$$

For $k \geq 3$. Since the $a(.), b(.)$ have the property that $\int_A^B a(t)b(t)\,dt = 0$ implies that $\int_A^B a(t)[b(t)]^{k-1} \neq 0$ for $k > 1$, (2.17) and (2.18) together imply that

$$\psi^{(k)}(0) = 0 \ \ for\ k \geq 3 \tag{2.19}$$

Which shows that $\psi(t) = i\lambda t - \frac{1}{2}\sigma^2 t^2$ where $-\infty < \lambda < \infty, \sigma^2 > 0$ for some λ and σ^2. Hence the process $\{X(t), t \in T\}$ is a Wiener process. This completes the proof of the "If" part.

References

[1] Cacoullos,T. (1967). Characterizations of normality by constant regression of linear statistics on another linear statistics. Ann. Math. Statist. 38, 1894 – 1898.

[2] Heyde, C.tC. (1970): Characterization of the normal law by the symmetry of certain conditional distribution. Sankhya Series A, 32, 115 – 118.

[3] Lukacs, E. (1965). Some results in the theory of the Wiener integrals. Trans Fourth. Prague. Conf., 29 – 43.

[4] Lukacs, E. (1968). Stochastic Convergence, D.C. Heath, Boston.

[5] Lukacs, E. and Laha, R.G. (1964), Applications of characteristic functions, Griffin, London.

[6] Mazo, J.E and Salz, J. (1970): A theorem o conditional expectation. IEEE Trans. On Information Theory Vol. IT – 16 , 379 – 381.

[7] Prakasa Rao, B.L.S (1970): On a Characterization of the Wiener Proecess by constant regression. Ann. Math. Stat. 41, 321 – 325.

[8] Rao, C.R. (1967). On some characterizations of the normal law. Sankhya Ser. A 29 1 – 14.

[9] Ramachandran, B. and Rao, C.R. (1970): Solutions of functional equations arising in some regression problems, and a characterization of the Cauchy Law. Sankhya, Series A, 32, 1 – 31.

[10] Wintner, A. (1947): The Fourier Transforms of Probability Distributions. Baltimore, Md.