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An Extension of Bilateral Generating Functions of the Biorthogonal Polynomials Suggestedby Laguerre Polynomials

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Abstract

In this note, we have obtained a novel extension of a bilateral generating relationsinvolvingbiorthogonal polynomials $Y_n^{\alpha}(x;k)$ from the existence of quasi-bilinear generating relation by group theoretic method. As particular cases, we obtain the corresponding results on generalised Laguerre polynomials.

Key words: Laguerre polynomials, Biorthogonal polynomials, Generating functions.

AMS-2010 Subject Classification Code: 33C 45, 33C47.

1. Introduction

The Konhauser biorthogonal polynomials $Y_n^{\alpha}(x;k)$[1] suggested by Laguerre polynomials[3] are defined byCarlitz[2] as follows:

$$Y_n^{\alpha}(x;k) = \frac{1}{n\,!}\sum_{i=0}^{n}\frac{x^i}{i\,!}\sum_{j=0}^{i}(-1)^j\binom{i}{j}\left(\frac{j+\alpha+1}{k}\right)_n,$$

where$(a)_n$ is the pochhammer symbol [4], $\alpha > -1,\ k$is a non-zero positive integer.

In [5], the quasi bilateral generating function is defined by

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n\, p_n^{(\alpha)}(x)\, q_m^{(n)}(u) w^n,$$

where$a_n$the coefficients are quite arbitrary and $p_n^{(\alpha)}(x)$, $q_m^{(n)}(u)$ are two special functions of orders $n\,,m$ and of parameters α and $n$respectively. If $q_m^{(n)}(u) \equiv p_m^{(n)}(u)$, the generating relation is known as quasi bilinear.

The aim at presenting this note is to prove the existence of a more general generating relation from the existence of a quasi-bilinear generating relation involving biorthogonal polynomials $Y_n^{\alpha}(x;k)$. In [6], Samanta and Chongdar have proved the following theorem on bilateral generating functions involving $Y_n^{\alpha}(x;k)$ by group-theoretic method.

Theorem 1 If there exists a unilateral generating relation of the form

$$G(x,w) = \sum_{n=0}^{\infty} a_n Y_n^{\alpha}(\mathrm{x};\mathrm{k})\, w^n \tag{1.1}$$

then

$$(1+kw)^{\frac{(1+\alpha-k)}{k}} \exp\left(x - x(1+kw)^{\frac{1}{k}}\right) G\left(x(1+kw)^{\frac{1}{k}},\ \ wv\right) = \sum_{n=0}^{\infty} w^n\, \sigma_n(x,v), \tag{1.2}$$

where

$$\sigma_n(x,v) = \sum_{p=0}^{n} a_p\, k^{n-p} \binom{n}{p} Y_n^{\alpha+pk-nk}(\mathrm{x};\mathrm{k})\, v^p.$$

The importance of the above theorem lies in the fact that whenever one knows a generating relation of the form (1.1) then the corresponding bilateral generating relation can at once be written down from (1.2). So one can get a large number of bilateral generating relations by attributing different suitable values to a_n in (1.1).

In the present paper, we have obtained the following extension (Theorem 2) of the theorem1 from the existence of quasi bilinear generating relation.

Theorem 2 If there exists a quasi-bilinear generating function of the following form

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n Y_n^{\alpha}(x;k)\; Y_m^{n}(u;k) w^n$$

then

$$(1+kw)^{\frac{(1+\alpha-k)}{k}} \exp\left(x - w - x(1+kw)^{\frac{1}{k}}\right) G\left(x(1+kw)^{\frac{1}{k}},\, u+w,\ wz\right)$$
$$= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} a_n \frac{w^{n+p+q}}{p!\,q!} (-1)^q (n+1)_p\, k^p\; Y_{n+p}^{\alpha-kp}(x;k) Y_m^{n+q}(u;k)\, z^n.$$

2. Proof of the theorem

For the biorthogonal polynomials, we consider the following operators:

$$R_1 = x\, y^{-k} z \frac{\partial}{\partial x} + y^{-(k-1)} z \frac{\partial}{\partial y} - (x + k - 1) y^{-k} z,$$

$$R_2 = v \frac{\partial}{\partial u} - v$$

such that

$$R_1(Y_n^\alpha(\mathrm{x};\mathrm{k}) y^\alpha z^n) = k(n+1) Y_{n+1}^{\alpha-k}(\mathrm{x};\mathrm{k}) y^{\alpha-k} z^{n+1}, \tag{2.1}$$

$$R_2(Y_m^n(\mathrm{u};\mathrm{k}) v^n) = -Y_m^{n+1}(\mathrm{x};\mathrm{k}) v^{n+1} \tag{2.2}$$

And

$$e^{wR_1} f(x,y,z) = (1 + kwy^{-k}z)^{\frac{1-k}{k}} \exp\left(x - x(1 + kwy^{-k}z)^{\frac{1}{k}}\right)$$
$$\times f\left(x(1 + kwy^{-k}z)^{\frac{1}{k}},\ y(1 + kwy^{-k}z)^{\frac{1}{k}},\ z\right) \tag{2.3}$$

$$e^{wR_2} f(u,v) = \exp(-wv) f(u + vw, v) \tag{2.4}$$

Let us now consider the generating relation of the form:

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n Y_n^\alpha(x;k) Y_m^n(u;k) w^n, \tag{2.5}$$

Replacing $w$by $wvz$and multiplying both sides of (2.5) by $y^\alpha$and then operating $e^{wR_1}\ e^{wR_2}$ on both sides, we get

$$e^{wR_1}\ e^{wR_2}[y^\alpha\ G(x,\ u, wvz)]$$
$$= e^{wR_1}\ e^{wR_2}\left[\sum_{n=0}^{\infty} a_n (Y_n^\alpha(x;k) y^\alpha z^n)(Y_m^n(u;k) v^n)\ w^n\right] \tag{2.6}$$

Now the left member of (2.6), with the help of (2.3) and (2.4), becomes

$$(1 + kwy^{-k}z)^{\frac{1+\alpha-k}{k}} \exp\left(x - wv - x(1 + kwy^{-k}z)^{\frac{1}{k}}\right) y^\alpha$$
$$\times G\left(x(1 + kwy^{-k}z)^{\frac{1}{k}}, u + vw, wvz\right). \tag{2.7}$$

The right member of (2.6), with the help of (2.1) and (2.2), becomes

$$\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} a_n \frac{w^{n+p+q}}{p!\,q!} (-1)^q (n + 1)_p k^p Y_{n+p}^{\alpha-kp}(x;k) Y_m^{n+q}(u;k) y^{\alpha-kp} z^{n+p} v^{n+q}. \tag{2.8}$$

Now equating both members, and the substituting $\frac{z}{y^k} = 1,\ \ v = 1$, we get

$$(1 + kw)^{\frac{1+\alpha-k}{k}} \exp\left(x - w - x(1 + kw)^{\frac{1}{k}}\right) G\left(x(1 + kw)^{\frac{1}{k}}, u + w, wz\right)$$
$$= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} a_n \frac{w^{n+p+q}}{p!\,q!} (-1)^q (n+1)_p k^p Y_{n+p}^{\alpha-kp}(x;k) Y_m^{n+q}(u;k) z^n. \tag{2.9}$$

This completes the proof of theorem 2.

Corollary: If we put $m=0$, we notice that $G(x,u,w)$ becomes $G(x,w)$ since $Y_0^{n+q}(u;k)=1$. Hence from (2.9), we get

$$(1+kw)^{\frac{1+\alpha-k}{k}}\exp\left(x-x(1+kw)^{\frac{1}{k}}\right)G\left(x(1+kw)^{\frac{1}{k}},\ wz\right)$$
$$=\sum_{n=0}^{\infty}\sum_{p=0}^{n}a_{n-p}\frac{w^n}{p!}(n-p+1)_p\,k^p\,Y_n^{\alpha-kp}(x;k)z^{n-p}.$$

Therefore we have

$$(1+kw)^{\frac{1+\alpha-k}{k}}\exp\left(x-x(1+kw)^{\frac{1}{k}}\right)G\left(x(1+kw)^{\frac{1}{k}},\ wz\right)=\sum_{n=0}^{\infty}w^n\,\sigma_n(x,z)$$

where

$$\sigma_n(x,z)=\sum_{p=0}^{n}a_p\binom{n}{p}\,k^{n-p}\,Y_n^{\alpha-nk+pk}(x;k)\,z^p,$$

which is theorem 1.

3. Special cases

We now proceed to find some special cases of our theorem 2.

Case 1 If we put $k=1$, then $Y_n^{\alpha}(x;k)$ reduces to generalised Laguerre polynomials $L_n^{\alpha}(x)$. Thus putting $k=1$ in our theorem, we get the following theorem on quasi-bilinear generating function involving modified Laguerre polynomials.

Theorem 3 If there exists a quasi-bilinear generating function of the following form

$$G(x,u,w)=\sum_{n=0}^{\infty}a_n\,L_n^{(\alpha)}(x)L_m^{(n)}(u)\,w^n \tag{3.1}$$

then

$$(1+w)^{\alpha}\exp\{-w(1+w)\}\,G(x(1+w),\,u+w,\ wz)$$
$$=\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}a_n\frac{w^{n+p+q}}{p!\,q!}(-1)^q(n+1)_p\,L_{n+p}^{\alpha-p}(x)L_m^{n+q}(u)\,z^n, \tag{3.2}$$

which is found derived in [7]

Case 2 putting $m=0$ in theorem 3 and then simplifying, we get the following theorem on bilateral generating functions involving the polynomials under consideration.

Theorem 4 If

$$G(x,w)=\sum_{n=0}^{\infty}a_n\,L_n^{(\alpha)}(x)\,w^n \tag{3.3}$$

Then

$$(1+w)^{\alpha} \exp(-wx)\, G(x(1+w),\ wz) = \sum_{n=0}^{\infty} w^n\, \sigma_n(x,z), \tag{3.4}$$

where

$$\sigma_n(x,z) = \sum_{p=0}^{n} a_p \binom{n}{p} L_n^{(\alpha-n+p)}(\mathrm{x})\, z^p,$$

which is found derived in [7,8,9,10]

Note: Here we would like to mention that the way, the authors of [8] used, to obtain the above result is incorrect. In fact, they multiplying both sides of the relation (3.1) of [8] by r^n to obtain the said result, which is inadmissible as n runs from 0to ∞.

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