International Journal of Mathematics Research.
ISSN 0976-5840 Volume 6, Number 1 (2014), pp. 15-26

http://www.irphouse.com

Approximation of Conjugate Function Belonging to $Lip(\xi(t), r)$ Class by (C,1) (E,1) Means

Hare Krishna Nigam

Department of Mathematics, Faculty of Engineering & Technology
Mody Institute of Technology and Science (Deemed University),
Laxmangarh, Sikar (Rajasthan), India.
E-mail:harekrishnan@yahoo.com

Abstract

In this paper, a new theorems on degree of approximation of a function $\overline{f}$, conjugate to a 2π periodic function f, belonging to $Lip(\xi(t), r)$ class by (C,1)(E,1) product means of conjugate Fourier series have been established.

2010 Mathematical Subject Classification: Primary 42B05, 42B08.

Key words and phrases: Degree of approximation, $Lip(\xi(t), r)$ class of function, (C,1) summability, (E,1) summability, (C,1)(E,1) product summability, Fourier series, conjugate Fourier series, Lebesgue integral.

1. Introduction:

Let f(x) be periodic with period 2π and integrable in the sense of Lebesgue. The Fourier series of f(x) is given by

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{1.1}$$

with n^{th} partial sums $s_n(f;x)$.

The conjugate series of the Fourier series (1.1) is given by

$$\sum_{n=1}^{\infty}(a_n \sin nx - b_n \cos nx) \tag{1.2}$$

with n^{th} partial sums $\overline{s_n}(f;x)$ and we shall call it as conjugate Fourier series through out the paper..

$L_\infty -$ norm of a function $f : R \to R$ is defined by $\|f\|_\infty = \sup\{|f(x)| : x \in R\}$

L_r – norm is defined by $\|f\|_r = \left(\int_0^{2\pi} |f(x)|^r dx \right)^{\frac{1}{r}}, r \geq 1$ (1.3)

The degree of approximation of a function $f : R \to R$ by a trigonometric polynomial t_n of order n under sup norm $\| \|_\infty$ is defined by Zygmund[13] and is given as

$$\|t_n - f\|_\infty = \sup\{|t_n(x) - f(x)| : x \in R\} \quad (1.4)$$

and $E_n(f)$ of a function $f \in L_r$ is given by

$$E_n(f) = \min\|t_n - f\|_r . \quad (1.5)$$

A function $f \in Lip\alpha$ if

$$f(x+t) - f(x) = O\left(|t|^\alpha\right) \text{ for } 0 < \alpha \leq 1 \quad (1.6)$$

$f(x) \in Lip(\alpha, r)$ if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^r dx \right)^{\frac{1}{r}} = O\left(|t|^\alpha\right),\ 0 < \alpha \leq 1, \text{ and } r \geq 1 \quad (1.7)$$

(definition 5.38 of Mc Fadden [7], 1942).

Given a positive increasing function $\xi(t)$ and an integer $r \geq 1$, $f(x) \in Lip(\xi(t), r)$ if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^r dx \right)^{\frac{1}{r}} = O(\xi(t)) \quad (1.8)$$

If $\xi(t) = t^\alpha$ then $Lip(\xi(t), r)$ class coincides with the class $Lip(\alpha, r)$ and if $r \to \infty$ then $Lip(\alpha, r)$ reduces to the class $Lip\alpha$.

Let $\sum_{n=0}^{\infty} u_n$ be a given infinite series with the sequence of its n[th] partial sums $\{s_n\}$.

The (C, 1) transform is defined as the n[th] partial sum of (C, 1) summability

$$t_n = \frac{s_0 + s_1 + s_2 + ... + s_n}{n+1}$$

$$= \frac{1}{n+1} \sum_{k=0}^{n} s_k \to s \ \ as \ \ n \to \infty \quad (1.10)$$

then the series $\sum_{n=0}^{\infty} u_n$ is summable to s by (C,1) method.

If $(E,1) = E_n^1 = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} s_k \to s \ \ as \ \ n \to \infty.$ (1.11)

then the infinite series $\sum_{n=0}^{\infty} u_n$ is said to be summable (E,1) to the definite number s (Hardy [3]).

The (C,1) transform of the (E,1) transform defines (C,1)(E,1) transform of the partial sum s_n of series $\sum_{n=0}^{\infty} u_n$ and we denote it by $(CE)_n^1$.

Thus if

$$(CE)_n^1 = \frac{1}{n+1}\sum_{k=0}^{n} E_k^1 \to s,\ as\ \ n\to\infty$$

$$= \frac{1}{n+1}\sum_{k=0}^{n}\left[\frac{1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}\right] \to s,\ as\ \ n\to\infty \tag{1.12}$$

where E_n^1 denotes the (E,1) transform of s_n and C_n^1 denotes (C,1) transform of s_n, then the series $\sum_{n=0}^{\infty} u_n$ is said to be summable by (C,1)(E,1) method or summable (C,1)(E,1) to a definite number s.

We use the following notations:

$$\psi(t) = f(x+t) + f(x-t)$$

$$\tau = \text{Integral part of } \frac{1}{t} = \left[\frac{1}{t}\right]$$

$$\overline{K}_n(t) = \frac{1}{2\pi(n+1)}\sum_{k=0}^{n}\left[\frac{1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}\frac{\cos\left(\nu+\frac{1}{2}\right)t}{\sin(t/2)}\right]$$

2. Main Theorems:

A good amount of work has been done on degree of approximation of functions belonging to $Lip\alpha$, $Lip(\alpha,r)$, $Lip(\xi(t),r)$ classes using Cesàro, Nörlund and generalised Nörlund single summability methods by a number of researchers like Alexits [1], Sahney and Goel [12], Qureshi and Neha [10], Quershi [8,9], Chandra [2], Khan [4], Leindler [6] and Rhoades [11]. But till now nothing seems to have been done so far in the direction of present work. Therefore, in present paper, a theorem on degree of approximation of a function $\overline{f}$, conjugate to a 2π periodic function f, belonging to $Lip(\xi(t),r)$ class using (C,1)(E,1) product summability means of conjugate Fourier series have been established in the following form:

2.1. Theorem.

If f is a 2π-periodic function, Lebesgue integrable on $[0,2\pi]$, belonging to the $Lip(\xi(t),r$ class then its degree of approximation by $\overline{(C,1)(E,1)}$ summability means of its conjugate Fourier series is given by

$$\left\|\overline{(CE)_n^1} - \overline{f}\right\|_r = O\left[(n+1)^{\frac{1}{r}}\xi\left(\frac{1}{(n+1)}\right)\right] \tag{2.1}$$

provided $\xi(t)$ satisfies the following conditions:

$$\left\{\int_0^{\frac{1}{n+1}}\left(\frac{t|\phi(t)|}{\xi(t)}\right)^r dt\right\}^{\frac{1}{r}} = O\left(\frac{1}{n+1}\right) \tag{2.2}$$

and

$$\left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{t^{-\delta}|\phi(t)|}{\xi(t)}\right)^r dt\right\}^{\frac{1}{r}} = O\left\{(n+1)^{\delta}\right\} \tag{2.3}$$

where δ is an arbitrary number such that $0 \neq \delta s + 1 < s$, $\frac{1}{r}+\frac{1}{s}=1$, conditions (2.2) and (2.3) hold uniformly in x and $\overline{(CE)_n^1}$ is $\overline{(C,1)(E,1)}$ means of the series (1.2) provided

$$2^{\tau}\sum_{k=\tau}^{n} 2^{-k} = O(n+1) \tag{2.4}$$

and

$$\overline{f}(x) = -\frac{1}{2\pi}\int_0^{\pi}\psi(t)\cot\frac{1}{2}t\, dt \tag{2.5}$$

3. Lemmas: For the proof of our theorem, following lemmas are required:

Lemma 1: $\overline{K}_n(t) = O\left(\frac{1}{t}\right)$ for $0 \le t \le \frac{1}{n+1}$

Proof: For $0 \le t \le \frac{1}{n+1}$, $\sin(t/2) \ge (t/\pi)$ and $|\cos nt| \le 1$

$$\left|\overline{K}_n(t)\right| = \frac{1}{2\pi(n+1)}\left|\sum_{k=0}^{n}\left[\frac{1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}\frac{\cos\left(\nu+\frac{1}{2}\right)t}{\sin(t/2)}\right]\right|$$

$$\le \frac{1}{2\pi(n+1)}\sum_{k=0}^{n}\frac{1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}\frac{\left|\cos\left(\nu+\frac{1}{2}\right)t\right|}{|\sin(t/2)|}$$

$$\le \frac{1}{2t(n+1)}\sum_{k=0}^{n}\frac{1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}$$

$$= \frac{1}{2t(n+1)}\sum_{k=0}^{n}\frac{1}{2^k}2^k$$

$$= \frac{1}{2t(n+1)}\sum_{k=0}^{n}1$$

$$= O\left(\frac{1}{t}\right)$$

Lemma 2: For $0 \le a \le b \le \infty,\ 0 \le t \le \pi$ and any n, we have

$$\overline{K}_n(t) = O\left(\frac{\tau^2}{(n+1)}\right) + O\left(\frac{\tau}{(n+1)}(1+q)^{\tau}\sum_{k=\tau}^{n}(1+q)^{-k}\right)$$

Proof: For $0 \le \frac{1}{n+1} \le t \le \pi,\ \sin(t/2) \ge (t/\pi)$

$$\left|\overline{K}_n(t)\right| = \frac{1}{2\pi(n+1)}\left|\sum_{k=0}^{n}\left[\frac{1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}\frac{\cos\left(\nu+\frac{1}{2}\right)t}{\sin(t/2)}\right]\right|$$

$$\le \frac{1}{2t(n+1)}\left|\sum_{k=0}^{n}\left[\frac{1}{2^k}\operatorname{Re}\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}\ e^{i\left(\nu+\frac{1}{2}\right)t}\right\}\right]\right|$$

$$\le \frac{1}{2t(n+1)}\left|\sum_{k=0}^{n}\left[\frac{1}{2^k}\operatorname{Re}\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}\ e^{i\nu t}\right\}\right]\right|\left|e^{\frac{it}{2}}\right|$$

$$\le \frac{1}{2t(n+1)}\left|\sum_{k=0}^{n}\left[\frac{1}{2^k}\operatorname{Re}\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}\ e^{i\nu t}\right\}\right]\right|$$

$$= \frac{1}{2t(n+1)}\left|\sum_{k=0}^{\tau-1}\left[\frac{1}{2^k}\operatorname{Re}\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}\ e^{i\nu t}\right\}\right]\right|$$

$$+\frac{1}{2t(n+1)}\left|\sum_{k=\tau}^{n}\left[\frac{1}{2^k}\operatorname{Re}\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}\ e^{i\nu t}\right\}\right]\right| \tag{3.1}$$

Now considering first term of (3.1)

$$\frac{1}{2t(n+1)}\left|\sum_{k=0}^{\tau-1}\left[\frac{1}{2^k}\operatorname{Re}\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}\ e^{i\nu t}\right\}\right]\right| \le \frac{1}{2t(n+1)}\left|\ \sum_{k=0}^{\tau-1}\frac{1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}\right|\left|e^{i\nu t}\right|$$

$$\le \frac{1}{2t(n+1)}\sum_{k=0}^{\tau-1}\left[\ \frac{1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}\ \right]$$

$$= \frac{1}{2t(n+1)}\sum_{k=0}^{\tau-1}1$$

$$= \frac{\tau}{2t(n+1)}$$

$$= O\left(\frac{\tau^2}{(n+1)}\right) \tag{3.2}$$

Now considering second term of (3.1) and using Abel's Lemma

$$\frac{1}{2t(n+1)}\left|\sum_{k=\tau}^{n}\left[\frac{1}{2^k}\operatorname{Re}\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}\ e^{i\nu t}\right\}\right]\right| \le \frac{1}{2t(n+1)}\sum_{k=\tau}^{n}\frac{1}{2^k}\max_{0\le m\le k}\left|\sum_{\nu=0}^{m}\binom{k}{\nu}e^{i\nu t}\right|$$

$$\le \frac{1}{2t(n+1)}2^{\tau}\sum_{k=\tau}^{n}\frac{1}{2^k}$$

$$= O\left[\frac{\tau}{(n+1)}2^{\tau}\sum_{k=\tau}^{n}\frac{1}{2^k}\right] \tag{3.3}$$

Combining (3.1), (3.2) and (3.3) we get,

$$\overline{K}_n(t) = O\left(\frac{\tau^2}{(n+1)}\right) + O\left(\frac{\tau}{(n+1)}2^{\tau}\sum_{k=\tau}^{n}2^{-k}\right) \tag{3.4}$$

4. Proof of the Theorem:

Let $\overline{s}_n(f;x)$ denote s, the n[th] partial sum of the series (1.2). Then following Lal[5], we have

$$\overline{s}_n(f;x) - \overline{f}(x) = \frac{1}{2\pi}\int_0^{\pi}\psi(t)\frac{\cos\left(n+\frac{1}{2}\right)t}{\sin\frac{t}{2}}dt$$

Therefore using (1.2), the (E,1) transform E_n^1 of $\overline{s}_n(f;x)$ is given by

$$\overline{E_n^1} - \overline{f}(x) = \frac{1}{2\pi\ 2^n}\int_0^{\pi}\frac{\psi(t)}{\sin(t/2)}\left\{\sum_{k=0}^{n}\binom{n}{k}\cos\left(k+\frac{1}{2}\right)t\right\}dt$$

Now denoting $\overline{(C,1)(E,1)}$ transform of $\overline{s}_n$ by $\overline{(CE)_n^1}$, we write

$$\overline{(CE)_n^1} - \overline{f}(x) = \frac{1}{2\pi(n+1)}\sum_{k=0}^{n}\left[\int_0^{\pi}\frac{\psi(t)}{\sin(t/2)}\ \frac{1}{2^k}\ \left\{\sum_{\nu=0}^{k}\binom{k}{\nu}\cos\left(\nu+\frac{1}{2}\right)t\right\}dt\right]$$

$$= \int_0^{\pi}\psi(t)\overline{K}_n(t)\,dt = \left[\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi}\right]\psi(t)\overline{K}_n(t)\,dt$$

$$= I_1 + I_2 \text{ (say)} \tag{4.1}$$

We consider,

$$|I_1| \le \int_0^{\frac{1}{n+1}}|\psi(t)|\ \left|\overline{K}_n(t)\right|\ dt$$

Using Hölder's inequality and the fact that $\psi(t) \in Lip(\xi(t), r)$,

$$\leq \left[\int_0^{\frac{1}{n+1}} \left\{\frac{t|\psi(t)|}{\xi(t)}\right\}^r dt\right]^{\frac{1}{r}} \left[\int_0^{\frac{1}{n+1}} \left\{\frac{\xi(t)\left|\overline{K}_n(t)\right|}{t}\right\}^s dt\right]^{\frac{1}{s}}$$

$$= O\left(\frac{1}{n+1}\right)\left[\int_0^{\frac{1}{n+1}} \left\{\frac{\xi(t)\left|\overline{K}_n(t)\right|}{t}\right\}^s dt\right]^{\frac{1}{s}} \qquad \text{by (2.2)}$$

$$= O\left(\frac{1}{n+1}\right)\left[\int_0^{\frac{1}{n+1}} \left\{\frac{\xi(t)}{t^2}\right\}^s dt\right]^{\frac{1}{s}} \qquad \text{by Leema 1}$$

Since $\xi(t)$ is a positive increasing function and using second mean value theorem for integrals,

$$I_1 = O\left\{\left(\frac{1}{n+1}\right)\xi\left(\frac{1}{n+1}\right)\right\}\left[\int_{\in}^{\frac{1}{n+1}} \frac{dt}{t^{2s}}\right]^{\frac{1}{s}} \quad \text{for some } 0 \leq \in < \frac{1}{n+1}$$

$$= O\left\{\left(\frac{1}{n+1}\right)\xi\left(\frac{1}{n+1}\right)\right\}\left[\left\{\frac{t^{-2s+1}}{-2s+1}\right\}_{\in}^{\frac{1}{n+1}}\right]^{\frac{1}{s}}$$

$$= O\left\{\left(\frac{1}{n+1}\right)\xi\left(\frac{1}{n+1}\right)\right\}\left\{(n+1)^{2-\frac{1}{s}}\right\}$$

$$= O\left\{(n+1)^{\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right\} \quad \because \frac{1}{r}+\frac{1}{s}=1 \qquad (4.2)$$

Using Hölder's inequality,

$$|I_2| \leq \int_{\frac{1}{n+1}}^{\pi} |\psi(t)|\left|\overline{K}_n(t)\right| dt$$

$$|I_{2.2}| = O\left[\int_{\frac{1}{n+1}}^{\pi} \frac{|\psi(t)|}{t^2(n+1)} dt\right] + O\left[\int_{\frac{1}{n+1}}^{\pi} \frac{|\psi(t)|}{t\ (n+1)} 2^{\tau} \sum_{k=\tau}^{n} 2^{-k} dt\right]$$

$$= O(I_{2.2.1}) + O(I_{2.2.2}) \text{ (say)} \qquad (4.3)$$

Using Hölder's inequality and Leema 5,

$$|I_{2.2.1}| \le \left(\frac{1}{n+1}\right)\left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{t^{-\delta}\,\psi(t)}{\xi(t)}\right\}^{r} dt\right]^{\frac{1}{r}}\left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{\xi(t)}{t^{2-\delta}}\right\}^{s} dt\right]^{\frac{1}{s}}$$

$$= O\left\{\frac{(n+1)^{\delta}}{n+1}\right\}\left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{\xi(t)}{t^{2-\delta}}\right\}^{s} dt\right]^{\frac{1}{s}} \text{ by (2.3)}$$

Now putting $t = \dfrac{1}{y}$,

$$I_{2.2.1} = O\left\{\frac{(n+1)^{\delta}}{n+1}\right\}\left[\int_{\frac{1}{\pi}}^{n+1}\left\{\frac{\xi\left(\frac{1}{y}\right)}{(y)^{\delta-2}}\right\}^{s}\frac{dy}{y^{2}}\right]^{\frac{1}{s}}$$

Since $\xi(t)$ is a positive increasing function and using second mean value theorem for integrals,

$$I_{2.2.1} = O\left\{\frac{(n+1)^{\delta}}{n+1}\,\xi\left(\frac{1}{n+1}\right)\right\}\left[\int_{\eta}^{n+1}\frac{dy}{y^{s(\delta-2)+2}}\right]^{\frac{1}{s}} \text{ for some } \frac{1}{\pi} \le \eta \le n+1$$

$$= O\left\{\frac{(n+1)^{\delta}}{n+1}\,\xi\left(\frac{1}{(n+1)}\right)\right\}\left[\int_{1}^{n+1}\frac{dy}{y^{s(\delta-2)+2}}\right]^{\frac{1}{s}} \text{ for } \frac{1}{\pi} < 1 \le n+1.$$

$$= O\left\{\frac{(n+1)^{\delta}}{n+1}\,\xi\left(\frac{1}{n+1}\right)\right\}\left[\left\{\frac{y^{s(2-\delta)-1}}{s(2-\delta)-1}\right\}_{1}^{n+1}\right]^{\frac{1}{s}}$$

$$= O\left\{\frac{(n+1)^{\delta}}{n+1}\,\xi\left(\frac{1}{n+1}\right)\right\}\left[(n+1)^{(2-\delta)-\frac{1}{s}}\right]$$

$$= O\left\{\xi\left(\frac{1}{n+1}\right)\right\}\left\{(n+1)^{1-\frac{1}{s}}\right\}$$

$$= O\left\{(n+1)^{\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right\} \because \frac{1}{r}+\frac{1}{s}=1$$

$$I_{2.2.1} = \mathrm{O}\left[\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right] \because \frac{1}{r}+\frac{1}{s}=1 \tag{4.4}$$

Similarly,

$$|I_{2.2.2}| \le \left(\frac{1}{n+1}\right)\left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{t^{-\delta}\,\psi(t)}{\xi(t)}\right\}^{r} dt\right]^{\frac{1}{r}} \left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{\xi(t)\,2^{\tau}\sum_{k=\tau}^{n}2^{-k}}{t^{1-\delta}}\right\}^{s} dt\right]^{\frac{1}{s}}$$

$$= O\left(\frac{1}{n+1}\right)\left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{t^{-\delta}\,\psi(t)}{\xi(t)}\right\}^{r} dt\right]^{\frac{1}{r}} \left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{\xi(t)\,(n+1)}{t^{1-\delta}}\right\}^{s} dt\right]^{\frac{1}{s}} \text{ by (2.4)}$$

$$= O\left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{t^{-\delta}\,\psi(t)}{\xi(t)}\right\}^{r} dt\right]^{\frac{1}{r}} \left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{\xi(t)}{t^{1-\delta}}\right\}^{s} dt\right]^{\frac{1}{s}}$$

$$= O\{(n+1)^{\delta}\}\left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{\xi(t)}{t^{1-\delta}}\right\}^{s} dt\right]^{\frac{1}{s}} \text{ by (2.3)}$$

Now putting $t = \frac{1}{y}$,

$$I_{2.2.2} = O\{(n+1)^{\delta}\}\left[\int_{\frac{1}{\pi}}^{n+1}\left\{\frac{\xi\left(\frac{1}{y}\right)}{(y)^{\delta-1}}\right\}^{s} \frac{dy}{y^{2}}\right]^{\frac{1}{s}}$$

Since $\xi(t)$ is a positive increasing function and using second mean value theorem for integrals,

$$I_{2.2.2} = O\left\{(n+1)^{\delta}\,\xi\left(\frac{1}{n+1}\right)\right\}\left[\int_{\eta}^{n+1}\frac{dy}{y^{s(\delta-1)+2}}\right]^{\frac{1}{s}} \text{ for some } \frac{1}{\pi} \le \eta \le n+1$$

$$= O\left\{(n+1)^{\delta}\,\xi\left(\frac{1}{(n+1)}\right)\right\}\left[\int_{1}^{n+1}\frac{dy}{y^{s(\delta-1)+2}}\right]^{\frac{1}{s}} \text{ for } \frac{1}{\pi} < 1 \le n+1.$$

$$= O\left\{(n+1)^{\delta}\,\xi\left(\frac{1}{n+1}\right)\right\}\left[\left\{\frac{y^{s(1-\delta)-1}}{s(1-\delta)-1}\right\}_{1}^{n+1}\right]^{\frac{1}{s}}$$

$$= O\left\{(n+1)^{\delta}\,\xi\left(\frac{1}{n+1}\right)\right\}\left[(n+1)^{(1-\delta)-\frac{1}{s}}\right]$$

$$= O\left\{\xi\left(\frac{1}{n+1}\right)\right\}\left\{(n+1)^{1-\frac{1}{s}}\right\}$$

$$= O\left\{(n+1)^{\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right\} \because \frac{1}{r}+\frac{1}{s}=1 \tag{4.5}$$

Combining (4.1) to (4.5),

$$\left|\overline{(CE)_n^1} - \bar{f}(x)\right| = O\left\{(n+1)^{\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right\}$$

Now using L_r - norm we get,

$$\left\|\overline{(CE)_n^1} - f(x)\right\|_r = \left\{\int_0^{2\pi}\left|\overline{(CE)_n^1} - f(x)\right|^r dx\right\}^{\frac{1}{r}}$$

$$= O\left[\left\{\int_0^{2\pi}\left\{(n+1)^{\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right\}^r dx\right\}^{\frac{1}{r}}\right]$$

$$= O\left\{(n+1)^{\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right\}\left[\left\{\int_0^{2\pi} dx\right\}^{\frac{1}{r}}\right]$$

$$= O\left\{(n+1)^{\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right\}$$

This completes the proof of theorem.

5. Applications:

The following corollaries can be derived from our main theorem:

Corollary 1: If $\xi(t) = t^\alpha$, $0 < \alpha \le 1$, then the class $Lip(\xi(t), r)$, $r \ge 1$, reduces to the class $Lip(\alpha, r)$ and the degree of approximation of a function $\bar{f}$, conjugate to a periodic function $f \in Lip(\alpha, r)$, $\frac{1}{r} < \alpha < 1$, is given by

$$\left|\overline{(CE)_n^1} - \bar{f}\right| = O\left(\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right)$$

Proof:

We have

$$\left\|\overline{(CE)_n^1} - \bar{f}\right\|_r = O\left\{\int_0^{2\pi}\left|\overline{(CE)_n^1} - \bar{f}\right|^r dx\right\}^{\frac{1}{r}}$$

or

$$\left\{(n+1)^{\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right\}=O\left\{\int_0^{2\pi}\left|\overline{(CE)_n^1}-\overline{f}\right|^r dx\right\}^{\frac{1}{r}}$$

or

$$O(1)=O\left\{\int_0^{2\pi}\left|\overline{(CE)_n^1}-\overline{f}\right|^r dx\right\}^{\frac{1}{r}}\ .\ O\left\{\frac{1}{(n+1)^{\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)}\right\}$$

Hence

$$\left|\overline{(CE)_n^1}-\overline{f}\right|=O\left\{(n+1)^{\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right\}$$

for if not the right-hand side will be O(1), therefore

$$\left|\overline{(CE)_n^1}-\overline{f}\right|=O\left\{\left(\frac{1}{n+1}\right)^{\alpha}(n+1)^{\frac{1}{r}}\right\}$$

$$=O\left(\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right)$$

Corollary 2: If $r\to\infty$ in corollary 1, then the class $Lip(\alpha,r)$ reduces to the $Lip\alpha$ class and the degree of approximation of a function $\overline{f}$, conjugate to a periodic function $f\in Lip\alpha$, $0<\alpha<1$ is given by

$$\left\|\overline{(CE)_n^1}-\overline{f}\right\|_\infty=O\left\{\frac{1}{(n+1)^{\alpha}}\right\}$$

Remark: An independent proof of above corollaries 1 can be obtained along the same lines of our theorem.

Acknowledgement:
Author is thankful to his parents for their encouragement and support to this work.

References:

[1] G. Alexits, Convergence problems of orthogonal series, Pergamon Press, London, 1961.
[2] Prem Chandra, Trigonometric approximation of functions in L_p- norm, J. Math. Anal. Appl. 275(2002), no. 1, (13-26).
[3] G. H. Hardy, Divergent series, first edition, Oxford University Press, 1949, 70.

[4] H. H. Khan, On degree of approximation of functions belonging to the class Lip(α, p), Indian J. Pure Appl. Math. 5(1974), no.2, 132-136.

[5] Shyam Lal, On K^{λ} summability of conjugate series of Fourier series, Bulletin of Calcutta Math. Soc., 89 (1997), 97-104.

[6] Lászaló Leindler,, Trigonometric approximation in L_p - norm, J. Math. Anal. Appl. 302(2005).

[7] Leonard McFadden, Absolute *Nöulund* summability, Duke Math. J. 9 (1942), 168-207.

[8] K. Qureshi, : On the degree of approximation of a periodic function f by almost *Nöulund* means, Tamkang J. Math. 12(1981), no. 1, 35-38.

[9] K. Qureshi, : On the degree of approximation of a function belonging to the class Lipα, Indian J. pure Appl. Math. 13(1982), no. 8, 560-563.

[10] K. Qureshi, and H. K. Neha, A class of functions and their degree of approximation, Ganita, 41(1) (1990) 37.

[11] B.E. Rhodes, On degree of approximation of functions belonging to lipschitz class (2003).

[12] B.N. Sahney, and D.S. Goel,, On the degree of continuous functions Ranchi, University Math. Jour., 4 (1973), 50-53.

[13] A. Zygmund, Trigonometric series, 2^{nd} rev. ed., Vol. 1, Cambridge Univ. Press, Cambridge, 1959.