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On Statistical Limit Points

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Abstract:-

The purpose of this paper is to study the statistical analogue of the set of limit points or cluster points of number sequences.

Key words:- Natural density, statistical convergence, statistical limit points, statistical cluster points.

1. Introduction:-

The idea of statistical convergence was introduced by Fast [3] and later on studied by several authors [1], [2], [4], [5], [6], [10], [13], [14]. In the whole paper IN stands for the set of natural numbers and we shall consider the sequences of real numbers.

The natural density [11], of a set A⊆IN is defined as $\delta(A)=\lim_{n\to\infty}\frac{1}{n}|\{a \in A: a \leq n\}|$
where the vertical bars indicates the number of elements in the enclosed set.

If $x=\{\xi_k\}_{k=1}^{\infty}$ is a sequence such that ξ_k satisfies property P for all k except for a set of natural density zero, then we write ξ_k satisfies P for " almost all k" (a.a.k.).

The sequence $x=\{\xi_k\}_{k=1}^{\infty}$ is said to converge statistically to the real number ξ (denoted by $st-lim\, x=\xi \;or\; \lim_{k\to\infty} stat\, \xi_k=\xi$) if for each $\varepsilon>o$ we have $\delta(A_\varepsilon)=0, where\; A_\varepsilon=\{n\in IN: |\xi_n-\xi|\geq\varepsilon\}$.

If $x=\{\xi_k\}$ is a sequence and if $\{\xi_{k(j)}\}$ is a subsequence of x and $K=\{k(j): j\in IN\}$ then we denote $\{\xi_{k(j)}\}$ by $\{x\}_K$. If $\delta(K)=0, \{x\}_K$ is called a subsequence of density zero or a thin subsequence. On the other hand $\{x\}_K$ is a non-thin subsequence of x if K does not have density zero. Moreover, it should be noted that $\{x\}_K$ is a non-thin subsequence of x if either $\delta(K)>0$ or $\delta(K)$ is not defined (i.e. K does not have natural density).

In [5] Fridy, introduced the concept of statistical limit points and statistical cluster points of real number sequences. Recall that the number λ is a statistical limit point of a sequence $x=\{\xi_k\}$ provided there is a non-thin subsequence of $x=\{\xi_k\}$ that converges to λ and a number γ is a statistical cluster point if a set $\{k\in IN: |\xi_k-\gamma|<\varepsilon\}$ does not have density zero for every $\varepsilon>0$. It was established that for a bounded sequence the set of statistical limit points may be empty while the set of statistical cluster points is non empty.

2. Statistical limit points and statistical cluster points

In this section we study the basic properties of statistical limit points and statistical cluster points. We also discuss the similarities and differences between these points and ordinary limit points.

- Definition (a): The number L is an ordinary limit point of a sequence x if there is a subsequence of x that converges to L.
- Definition (b): The number λ is a statistical limit point of a sequence x if there is a non-thin subsequence of x that converges to λ.

For any sequence $x, let\ \wedge_x\ and\ L_x$ denotes the set of statistical limit points and the set of ordinary limit points respectively.

Let us consider a sequence x= $\{\xi_k\}_{k=1}^{\infty}$

$\xi_k = \begin{cases} 1: if\ k\ is\ a\ square \\ 0: otherwise, \end{cases}$

then $L_x = \{0, 1\}\ and\ \wedge_x = \{0\}$. So, we have $\wedge_x \subseteq L_x$. But $\wedge_x$ and L_x can be very different for this let us consider another sequence, $x = \{r_k\}_{k=1}^{\infty}$ whose range is the set of all rational numbers and define $x = \{\xi_k\}_{k=1}^{\infty}$ by $\xi_k = \begin{cases} r_n \text{ ;if } k=n^2\ for\ n=1,2,3,... \\ k \text{ ; otherwise.} \end{cases}$

Since the set of squares have density zero it follows, $\Lambda_x = \phi$. While, the fact the every real number is a limit point of $\{r_k : k \in IN\}$ implies that $L_x = \mathrm{I}R$ (i.e. set of real numbers).

- Definition (c): The number γ is a statistical cluster point of a sequence x if for every $\varepsilon > 0$, the set {k ϵ IN:$|\xi_k - \gamma| < \varepsilon\}$ does not have density zero.

For a given sequence x, let Γ_x denotes the set of all statistical cluster points of x. It is clear that $\Gamma_x \subseteq L_x$ for every sequence x.

3. Main Results:

The following proposition gives the inclusion relation between $\Gamma_x\ and\ \Lambda_x$.

Proposition 1. For any number sequence $x, \Lambda_x \subseteq \Gamma_x$.

Proof: Suppose $\lambda \in \Lambda_x$. Then $\exists$ a non-thin subsequence of $x = \{\xi_k\}_{k=1}^{\infty}$ say $\{\xi_{k(j)}\}$ that converges to λ and $\delta(K) = d > 0$, where K={k(1), k(2), k(3), …}. As, $lim\ \xi_{k(j)} = \lambda$, so for each $\varepsilon > 0\ \exists$ a positive integer j_o, such that $\left|\xi_{k(j)} - \lambda\right| < \varepsilon\ \forall\ j \geq j_o$. Hence for each $\varepsilon > 0$, { j: $\left|\xi_{k(j)} - \lambda\right| \geq \varepsilon\}$ is a finite set and so, $\left\{k \in IN: \left|\xi_k - \lambda\right| < \varepsilon\right\} \supseteq \{k(j): j \in IN\} \sim \{\text{finite set}\}$.

Therefore, $\frac{1}{n}\left|\{k \leq n: \left|\xi_k - \lambda\right| < \varepsilon\}\right| \geq \frac{1}{n}|\{k(j) \leq n\}| - \frac{1}{n}0(1)$ and this implies $\lim_{n\to\infty} \frac{1}{n}|\{\ k \leq n: |\xi_k - \lambda| < \varepsilon\}| \geq d$. Hence $\delta\{k \in IN: \left|\xi_k - \lambda\right| < \varepsilon\} \neq 0$. Thus λ is a statistical cluster point of x. Hence $\Lambda_x \subseteq \Gamma_x$.

The completes the proof.

Athough our experience with ordinary limit points may lead us to expect that Λ_x and Γ_x are equivalent. The next example shows that this is not always the case.

Example 2: Define a sequence $x = \{\xi_k\}\, by\, \xi_k = \frac{1}{p}$, where $k = 2^{p-1}(2q + 1); i.e\ (p-1)$ is the number of factors of 2 in the prime factorization of k.

It is easy to see that for each p, $\delta\left\{\mathrm{k}: \xi_{\mathrm{k}} = \frac{1}{\mathrm{p}}\right\} = 2^{-\mathrm{p}} > 0$. Thus $\frac{1}{p} \epsilon\, \Lambda_{\mathrm{x}}$. Also, $\delta\left\{\mathrm{k}: 0 < \xi_{\mathrm{k}} < \frac{1}{\mathrm{p}}\right\} = 2^{-\mathrm{p}}$ so, $0\, \epsilon\, \Gamma_{\mathrm{x}}$. Hence, $\Gamma_{\mathrm{x}} = \{0\} \cup \{\frac{1}{\mathrm{p}}\}$. But $0 \notin \Lambda_{\mathrm{x}}$ for if, $\{x\}_K$ is a subsequence that has limit zero then we have $\delta(\mathrm{K}) = 0$.

Proposition 3: If x is a statistically convergent sequence say $st - \lim\, x = \lambda$ then Λ_x and Γ_x are both equal to singleton set $\{\lambda\}$.

Proof:- As $st - \lim\, x = \lambda, so\ for\ every\ \varepsilon > 0,$

$\lim_{n\to\infty} \frac{1}{n} \left|\left\{\mathrm{k} \le \mathrm{n}: \left|\xi_{\mathrm{k}} - \lambda\right| \ge \varepsilon\right\}\right| = 0.$

This implies $\delta(\mathrm{A}_\varepsilon) = 0$ where $\mathrm{A}_\varepsilon = \left\{n \in IN: \left|\xi_{\mathrm{n}} - \lambda\right| \ge \varepsilon\right\}$ and so $\delta(A_\varepsilon^c) = 1$ where $A_\varepsilon^c = \{n \in IN: |\xi_{\mathrm{n}} - \lambda| < \varepsilon\}$ and thus $\delta\{n \in IN: |\xi_{\mathrm{n}} - \lambda| < \varepsilon\} \neq 0$.

Therefore, $\lambda \in \Gamma_{\mathrm{x}}$. Similarly, we can show that $\lambda \in \Lambda_{\mathrm{x}}$. Further, it can be proved that λ is the only statistical limit point and statistical cluster point of the sequence x.

This completes the proof..

The converse of the above proposition need not hold. For this consider the following example.

Example 4: Consider the sequence $x = \{\xi_k\}$ defined by $\xi_k = \{1 + (-1)^k\}k$. We have, $\Lambda_x = \{0\}, \Gamma_x = \{0\}$. But $x = \{\xi_k\}_{k=1}^\infty$ does not converges statistically to zero.

From example 2, we see that Λ_x need be a closed point set, but the next result shows that Γ_x like L_x is always a closed set.

Propositions 5: For any sequence x, the set Γ_x of statistical cluster points of x is a closed point set.

Proof: Let p be an accumulation point of Γ_x.Then for every $\varepsilon > o, \Gamma_x$ contains some point γ in$(p - \varepsilon, p + \varepsilon)$. Choose ε' so that,

$(\gamma - \varepsilon', \gamma + \varepsilon') \subseteq (p - \varepsilon, p + \varepsilon)$.

Since $\gamma\, \epsilon\, \Gamma_x, so, \delta\{\mathrm{k}: |\xi_{\mathrm{k}} - \gamma| < \epsilon'\} \neq 0$,

i.e. $\delta\{\, \mathrm{k}: \xi_{\mathrm{k}} \epsilon (\gamma - \varepsilon', \gamma + \varepsilon')\} \neq 0$.

Therefore, $\delta\{\, \mathrm{k}: \xi_{\mathrm{k}} \epsilon (p - \varepsilon, p + \varepsilon)\} \neq 0$.

Hence $p \in \Gamma_x\ and\ \Gamma_x\ is\ a\ closed\ point\ set.$

This completes the proof.

Theorem 6: If $x = \{\xi_k\}_{k=1}^\infty$ and $y = \{\eta_k\}_{k=1}^\infty$ are sequences such that $\xi_k = \eta_k a.a.k.$, then $\Lambda_{\mathrm{x}} = \Lambda_{\mathrm{y}}$ and $\Gamma_x = \Gamma_y$.

Proof: As $\xi_k = \eta_k a.a.k.$, therefore, $\delta\{\mathrm{k}: \xi_{\mathrm{k}} \neq \eta_{\mathrm{k}}\} = 0$. Let $\lambda \in \Lambda_{\mathrm{x}}$, so $\exists$ a non thin subsequence of x say $\{x\}_K$ that converges to λ. Since $\delta\{\mathrm{k}: \mathrm{k} \in \mathrm{K}\, \mathrm{and}\, \xi_{\mathrm{k}} \neq \eta_{\mathrm{k}}\} = 0$, it follows that the $\{k: k \in K\ and\ \xi_k = \eta_k\}$ does not have density zero. Therefore, the set $\{k: k \in K \; and \; \xi_k = \eta_k\}$ yields a non-thin subsequence $\{y\}_{K'} of \; \{y\}_K$ that converges to λ. Hence $\lambda \in \Lambda_y$. This implies $\Lambda_x \subseteq \Lambda_y$.

By symmetry, we see that $\Lambda_y \subseteq \Lambda_x$.

Therefore, $\Lambda_x = \Lambda_y$.

The assertion that $\Gamma_x = \Gamma_y$ can be proved by a similar argument.

This completes the proof.

Following theorem establishes a connection between statistical cluster points and ordinary limit points.

Theorem 7: If $x = \{\xi_k\}$ is a number sequence then $\exists$ a sequence $y = \{\eta_k\} \; s.t. L_y = \Gamma_x$ and $\eta_k = \xi_k \; a.a.k$,. moreover, the range of y is a subset of the range of x.

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