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Oscillatory Behavior of Fractional Difference Equations

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ABSTRACT

In this paper, we study oscillatory behavior of the fractional difference equations of the following form

$$\Delta(p(t)g(\Delta^{\alpha}x(t)))+q(t)f\left(\sum_{s=t_0}^{t-1+\alpha}(t-s-1)^{(-\alpha)}x(s)\right)=0,\ t\in N_{t_0+1-\alpha},$$

where Δ^{α} denotes the Riemann-Liouville difference operator of order α, $0<\alpha\leq 1$. We establish some oscillation criteria for the equation using Riccati transformation technique and Hardy inequality. Examples are provided to illustrate our main results.

1. INTRODUCTION

Oscillatory behavior of fractional differential equations have been investigated by few authors, see papers [2]-[8] and the theory of fractional differential equations are presented in the books, see [13]-[15]. But the fractional difference equations are studied by very few authors, see [9]-[12]. Motivated by [3] and [8], we study the following fractional difference equation of the form

$$\Delta(p(t)g(\Delta^{\alpha}x(t)))+q(t)f\left(\sum_{s=t_0}^{t-1+\alpha}(t-s-1)^{(-\alpha)}x(s)\right)=0,\ t\in N_{t_0+1-\alpha},\tag{1}$$

where Δ^{α} denotes the Riemann-Liouville difference operator of order $0<\alpha\leq 1$.

In this paper, we make the following assumptions.

(H_1). $p(t)$ and $q(t)$ are positive sequences and $f,g:R\to R$ are continuous functions with $xf(x)>0$, $xg(x)>0$ for $x\neq 0$ and there exist positive constants k_1, k_2 such that $\frac{f(x)}{x}\geq k_1$, $\frac{x}{g(x)}\geq k_2$ for all $x\neq 0$.

(H_2). $g^{-1}\in C(R,R)$ is a continuous function with $x\,g^{-1}(x)>0$ for $x\neq 0$ and there exists some positive constant ν_1 such that $g^{-1}(xy) \geq \nu_1 g^{-1}(x) g^{-1}(y)$.

A solution *x(t)* of (1) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is nonoscillatory. Equation (1) is said to be oscillatory if all its solutions are oscillatory.

2. PRELIMINARIES AND BASIC LEMMAS

In this section, we introduce some preliminary results of discrete fractional calculus, which will be used throughout this paper.

Definition 2.1. (see [11]) Let $\nu > 0$. The ν - *th* fractional sum of f is defined by

$$\Delta^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{(\nu-1)} f(s),$$

where f is defined for $s = a$ mod (1) and $\Delta^{-\nu} f$ is defined for $t = (a+\nu) \bmod (1)$ and $t^{(\nu)} = \dfrac{\Gamma(t+1)}{\Gamma(t-\nu+1)}$. The fractional sum $\Delta^{-\nu} f$ maps functions defined on N_a to functions defined on $N_{a+\nu}$.

Definition 2.2. (see [11]) Let $\mu > 0$ and $m-1 < \mu < m$, where m denotes a positive integer, $m = \lceil \mu \rceil$. Set $\nu = m - \mu$. The μ *-th* fractional difference is defined as

$$\Delta^{\mu} f(t) = \Delta^{m-\nu} f(t) = \Delta^{m} \Delta^{-\nu} f(t).$$

Lemma 2.3. Let *x(t)* be a solution of (1) and let

$$G(t) = \sum_{s=t_0}^{t-1+\alpha} (t-s-1)^{(-\alpha)} x(s) \tag{2}$$

Then

$$\Delta(G(t)) = \Gamma(1-\alpha) \Delta^{\alpha}(x(t)). \tag{3}$$

Proof:

$$G(t) = \sum_{s=t_0}^{t-1+\alpha} (t-s-1)^{(-\alpha)} x(s)$$

$$= \sum_{s=t_0}^{t-(1-\alpha)} (t-s-1)^{(1-\alpha)-1} x(s)$$

$$= \Gamma(1-\alpha) \Delta^{-(1-\alpha)} x(t),$$

which implies

$$\Delta(G(t)) = \Gamma(1-\alpha) \Delta \Delta^{-(1-\alpha)} x(t) = \Gamma(1-\alpha) \Delta^{\alpha} x(t).$$

In order to discuss our results in Section 3, now, we state the following lemma.

Lemma 2.4. (Hardy et al. see [1]) If X and Y are nonnegative, then

$$mXY^{m-1}-X^{m}\le(m-1)Y^{m} \quad \text{for } m>1 \tag{4}$$

where equality holds if and only if X = Y.

3. MAIN RESULTS

Theorem 3.1. Suppose that (H_1) − (H_2) and

$$\sum_{s=t_0}^{\infty} g^{-1}\left(\frac{1}{p(s)}\right)=\infty \tag{5}$$

hold. Furthermore, assume that there exists a positive sequence *b(t)* such that

$$\limsup_{t\to\infty}\sum_{s=t_0}^{t-1}\left(k_1 b(s)q(s)-\frac{(\Delta b_+(s))^2}{4b^2(s+1)R(s)}\right)=\infty, \tag{6}$$

where $R(t)=\dfrac{k_2 v_1 g^{-1}\left(\dfrac{p(t+1)}{p(t)}\right)b(t)\Gamma(1-\alpha)}{b^2(t+1)p(t+1)}$ and $\Delta b_+(s)=\max\left[\Delta b(s),0\right]$. Then every solution of (1) is oscillatory.

Proof. Suppose the contrary that *x(t)* is a nonoscillatory solution of (1). Without loss of generality, we may assume that *x(t)* is an eventually positive solution of (1). Then there exists $t_1\ge t_0$ such that

$$x(t)>0 \ \text{ and } \ G(t)>0 \ \text{ for } \ t\ge t_1 \tag{7}$$

where *G* is defined as in (2). Therefore, it follows from (1) that

$$\Delta(p(t)g(\Delta^{\alpha}x(t)))=-q(t)f(G(t))<0 \quad \text{for} \quad t\ge t_1. \tag{8}$$

Thus $p(t)g(\Delta^{\alpha}x(t))$ is an eventually non increasing sequence. First we show that $p(t)g(\Delta^{\alpha}x(t))$ is eventually positive. Suppose there is an integer $t_1> t_0$ such that $p(t_1)g(\Delta^{\alpha}x(t_1))=c<0$ for $t\ge t_1$ so that

$$p(t)g(\Delta^{\alpha}x(t))\le p(t_1)g(\Delta^{\alpha}x(t_1))=c<0$$

$$\Delta^{\alpha}x(t)\le g^{-1}\left\{\frac{c}{p(t)}\right\}<0$$

which implies that

$$\frac{\Delta G(t)}{\Gamma(1-\alpha)}=\Delta^{\alpha}x(t)\le v_1 g^{-1}(c)g^{-1}\left\{\frac{1}{p(t)}\right\} \ \text{ for } \ t\ge t_1.$$

Summing both sides of the last inequality from t_1 to $t-1$, we get

$$G(t)\le G(t_1)+v_1\Gamma(1-\alpha)\,g^{-1}(c)\sum_{s=t_1}^{t-1}g^{-1}\left\{\frac{1}{p(s)}\right\}\to-\infty \ \text{ as } \ t\to\infty, \tag{9}$$

which contradicts the fact that $G(t)>0$. Hence $p(t)g(\Delta^{\alpha}x(t))>0$ eventually. Define the function $\omega(t)$ by the Riccati substitution

$$\omega(t)=b(t)\frac{p(t)g(\Delta^{\alpha}x(t))}{G(t)} \quad \text{for } t\geq t_1. \tag{10}$$

Then we have $\omega(t)>0$ for $t\geq t_1$. It follows that

$$\Delta\omega(t)=\Delta b(t)\frac{\omega(t+1)}{b(t+1)}+\frac{b(t)\Delta(p(t)g(\Delta^{\alpha}x(t)))G(t+1)-b(t)p(t+1)g(\Delta^{\alpha}x(t+1))\Delta G(t)}{G(t+1)G(t)}$$

$$\leq\Delta b_{+}(t)\frac{\omega(t+1)}{b(t+1)}-\frac{b(t)q(t)f(G(t))}{G(t)}-\frac{b(t)p(t+1)g(\Delta^{\alpha}x(t+1))\Delta G(t)}{G^{2}(t+1)}$$

We have

$$p(t)g(\Delta^{\alpha}x(t))\geq p(t+1)g(\Delta^{\alpha}x(t+1))$$

$$\Delta^{\alpha}x(t)\geq\nu_1 g^{-1}\left(\frac{p(t+1)}{p(t)}\right)\Delta^{\alpha}x(t+1).$$

Using the above inequality

$$\Delta\omega(t)\leq\Delta b_{+}(t)\frac{\omega(t+1)}{b(t+1)}-k_1 b(t)q(t)-\frac{b(t)p(t+1)g(\Delta^{\alpha}x(t+1))\Delta G(t)}{G^{2}(t+1)}$$

$$\leq\Delta b_{+}(t)\frac{\omega(t+1)}{b(t+1)}-k_1 b(t)q(t)-\frac{b(t)p(t+1)g(\Delta^{\alpha}x(t+1))\Delta G(t)}{\frac{b^{2}(t+1)p^{2}(t+1)g(\Delta^{\alpha}x(t+1))^{2}}{\omega(t+1)^{2}}}$$

$$\leq\Delta b_{+}(t)\frac{\omega(t+1)}{b(t+1)}-k_1 b(t)q(t)-\frac{b(t)\Gamma(1-\alpha)\nu_1 g^{-1}\left(\frac{p(t+1)}{p(t)}\right)\Delta^{\alpha}x(t+1)}{\frac{b^{2}(t+1)p(t+1)g(\Delta^{\alpha}x(t+1))}{\omega(t+1)^{2}}}$$

$$\leq\Delta b_{+}(t)\frac{\omega(t+1)}{b(t+1)}-k_1 b(t)q(t)-\frac{k_2\nu_1\Gamma(1-\alpha)g^{-1}\left(\frac{p(t+1)}{p(t)}\right)b(t)}{b^{2}(t+1)p(t+1)}\omega(t+1)^{2} \tag{11}$$

$$=\Delta b_{+}(t)\frac{\omega(t+1)}{b(t+1)}-k_1 b(t)q(t)-R(t)\omega(t+1)^{2}$$

Where

$R(t)=\dfrac{k_2\nu_1 g^{-1}\left(\frac{p(t+1)}{p(t)}\right)b(t)\Gamma(1-\alpha)}{b^{2}(t+1)p(t+1)}$. Take $m=2$, $X=\sqrt{R(t)}\,\omega(t+1)$ and $Y=\dfrac{\Delta b_{+}(t)}{2b(t+1)\sqrt{R(t)}}$

Using the inequality (4), we have

$$2\left(\sqrt{R(t)}\omega(t+1)\right)\left(\frac{\Delta b_+(t)}{2b(t+1)\sqrt{R(t)}}\right)^{(2-1)}-\left(\sqrt{R(t)}\omega(t+1)\right)^2$$

$$\leq(2-1)\left(\frac{\Delta b_+(t)}{2b(t+1)\sqrt{R(t)}}\right)^2$$

$$=\frac{(\Delta b_+(t))^2}{4b^2(t+1)R(t)}$$

From (11), we conclude that

$$\Delta\omega(t)\leq-k_1b(t)q(t)+\frac{(\Delta b_+(t))^2}{4b^2(t+1)R(t)}.$$

Summing the above inequality from t_1 to $t-1$, we have

$$\sum_{s=t_1}^{t-1}\left(k_1b(s)q(s)-\frac{\Delta^2 b_+(s)}{4b^2(s+1)R(s)}\right)\leq\omega(t_1)-\omega(t)\leq\omega(t_1)<\infty,\quad\text{for}\quad t\geq t_1$$

Letting t → ∞, we get

$$\limsup_{t\to\infty}\sum_{s=t_1}^{t-1}\left(k_1b(s)q(s)-\frac{\Delta^2 b_+(s)}{4b^2(s+1)R(s)}\right)\leq\omega(t_1)<\infty,$$

which contradicts (6). The proof is complete.

Theorem 3.2. Suppose that (H_1) to (H_2) and $\sum_{s=t_0}^{\infty}p^{-1/\gamma}(s)=\infty$ hold. Furthermore, assume that there exists a positive sequence b(t) such that

$H(t,t)=0$ for $t\geq 0$ $H(t,s)>0$ $t>s\geq 0$

$\Delta_2 H(t,s)=H(t,s+1)-H(t,s)\leq 0$ for $t\geq s\geq 0$.

If

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\sum_{s=t_0}^{t-1}\left(b(s)q(s)H(t,s)-\frac{h_+^2(t,s)}{4k_1H(t,s)R(s)}\right)=\infty \tag{12}$$

where $h_+(t,s)=\Delta_2 H(t,s)+\dfrac{H(t,s)\Delta b_+(s)}{b(s+1)}$ and $\Delta b_+(s)=\max[\Delta b(s),0]$. Then every solution of (1) is oscillatory.

Proof. Suppose the contrary that *x(t)* is a non oscillatory solution of (1). Without loss of generality, we may assume that *x(t)* is an eventually positive solution of (1). We proceed as in the proof of Theorem (3.1). Multiplying (11) by H(t, s) and summing from t_1 to $t-1$, we obtain

$$\sum_{s=t_1}^{t-1} k_1 b(s)q(s)H(t,s) \le -\sum_{s=t_1}^{t-1} H(t,s)\Delta\omega(s) + \sum_{s=t_1}^{t-1} H(t,s)\Delta b_+(t)\frac{\omega(s+1)}{b(s+1)} - \sum_{s=t_1}^{t-1} H(t,s)R(s)\omega^2(s+1) \tag{13}$$

Using the summation by parts formula, we obtain

$$\begin{aligned}-\sum_{s=t_1}^{t-1} H(t,s)\Delta\omega(s) &= -\left[H(t,s)\omega(s)\right]_{s=t_1}^{t} + \sum_{s=t_1}^{t-1}\omega(s+1)\Delta_2 H(t,s)\\ &= H(t,t_1)\omega(t_1) + \sum_{s=t_1}^{t-1}\omega(s+1)\Delta_2 H(t,s)\end{aligned} \tag{14}$$

where $\Delta_2 H(t,s) = H(t,s+1) - H(t,s)$. For $t \ge t_1$ we have

$$k_1\sum_{s=t_1}^{t-1} b(s)q(s)H(t,s) \le H(t,t_1)\omega(t_1) + \sum_{s=t_1}^{t-1}\omega(s+1)\Delta_2 H(t,s) + \sum_{s=t_1}^{t-1} H(t,s)\Delta b_+(t)\frac{\omega(s+1)}{b(s+1)} - \sum_{s=t_1}^{t-1} H(t,s)R(s)\omega^2(s+1)$$

$$\begin{aligned}k_1\sum_{s=t_1}^{t-1} b(s)q(s)H(t,s) &\le H(t,t_1)\omega(t_1) + \sum_{s=t_1}^{t-1}\left(\Delta_2 H(t,s) + \frac{H(t,s)\Delta b_+(t)}{b(s+1)}\right)\omega(s+1) - \sum_{s=t_1}^{t-1} H(t,s)R(s)\omega^2(s+1)\\ &\le H(t,t_1)\omega(t_1) + \sum_{s=t_1}^{t-1}\left(h_+(t,s)\omega(s+1) - H(t,s)R(s)\omega^2(s+1)\right)\end{aligned} \tag{15}$$

where $h_+(t,s) = \Delta_2 H(t,s) + \dfrac{H(t,s)\Delta b_+(t)}{b(s+1)}$ is defined as in Theorem 3.2. Take m=2, $X = \sqrt{H(t,s)R(s)}\,\omega(s+1)$ and $Y = \dfrac{h_+(t,s)}{2\sqrt{H(t,s)R(s)}}$ and using the Lemma 2.4 see [1]

$$2\left(\sqrt{H(t,s)R(s)}\,\omega(s+1)\right)\left(\frac{h_+(t,s)}{2\sqrt{H(t,s)R(s)}}\right)^{(2-1)} - \left(\sqrt{H(t,s)R(s)}\,\omega(s+1)\right)^2$$

$$\le (2-1)\left(\frac{h_+(t,s)}{2\sqrt{H(t,s)R(s)}}\right)^2$$

$$= \left(\frac{h_+^2(t,s)}{4\sqrt{H(t,s)R(s)}}\right)$$

From equation (14), we have

$\Delta_2 H(t,s) \le 0$ for $t > s \ge t_0$, $0 < H(t,t_1) \le H(t,t_0)$ for $t > t_1 \ge t_0$

$$\sum_{s=t_1}^{t-1} b(s)q(s)H(t,s) \le k_1^{-1}H(t,t_1)\omega(t_1) + \sum_{s=t_1}^{t-1} \frac{h_+^2(t,s)}{4k_1H(t,s)R(s)}$$

$$\sum_{s=t_1}^{t-1}\left(b(s)q(s)H(t,s) - \frac{h_+^2(t,s)}{4k_1H(t,s)R(s)} \right) \le k_1^{-1}H(t,t_1)\omega(t_1)$$

$$\le k_1^{-1}H(t,t_0)\omega(t_1).$$

Since $0 < H(t,s) \le H(t,t_0)$ for $t > s \ge t_0$, we have $0 < \dfrac{H(t,s)}{H(t,t_0)} \le 1$ for $t > s \ge t_0$. Hence it follows from that

$$\frac{1}{H(t,t_0)} \sum_{s=t_0}^{t-1}\left(b(s)q(s)H(t,s) - \frac{h_+^2(t,s)}{4k_1H(t,s)R(s)} \right)$$

$$= \frac{1}{H(t,t_0)} \sum_{s=t_0}^{t-1}\left(b(s)q(s)H(t,s) - \frac{h_+^2(t,s)}{4k_1H(t,s)R(s)} \right)$$

$$+ \frac{1}{H(t,t_0)} \sum_{s=t_1}^{t-1}\left(b(s)q(s)H(t,s) - \frac{h_+^2(t,s)}{4k_1H(t,s)R(s)} \right)$$

$$\le \frac{1}{H(t,t_0)} \sum_{s=t_0}^{t_1-1} b(s)q(s)H(t,s) + \frac{1}{H(t,t_0)} k_1^{-1}H(t,t_0)\omega(t_1)$$

$$\le \sum_{s=t_0}^{t_1-1} b(s)q(s) + k_1^{-1}\omega(t_1)$$

Letting t → ∞, we have

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \sum_{s=t_0}^{t-1}\left(b(s)q(s)H(t,s) - \frac{h_+^2(t,s)}{4k_1H(t,s)R(s)} \right) \le \sum_{s=t_0}^{t_1-1} b(s)q(s) + k_1^{-1}\omega(t_1) < \infty$$

which is a contradiction to (11). The proof is complete.

Example 3.3. Consider the following fractional difference equation

$$\Delta\left(t^{1/3}\left(\Delta^{1/2}\left(x(t)\right)\right)\right) + t\left(\sum_{s=t_0}^{t-1+\alpha} (t-s-1)^{(-\alpha)}x(s)\right) = 0,\ t > 0 \tag{16}$$

where $\alpha = 1/2$, $p(t) = t^{1/3}$, $q(t) = t$ and $f(x) = g(x) = x$. Take $k_1 = k_2 = 1$, $\nu_1 = 1$. Since

$$\sum_{s=t_0}^{\infty} \frac{1}{p(s)} = \sum_{s=t_0}^{\infty} \frac{1}{s^{\frac{1}{3}}} = \infty \tag{17}$$

we find that (H_1) − (H_2) and (5) hold. We will apply Theorem (3.1) and it remains to show that condition (6) is satisfied. Taking *b(s) = s*, we obtain

$$\limsup_{t\to\infty} \sum_{s=t_0}^{t-1}\left(k_1b(s)q(s) - \frac{\Delta b_+(s)}{4b^2(s+1)R(s)} \right) = \limsup_{t\to\infty} \sum_{s=t_0}^{t-1}\left(s^2 - \frac{s^{1/3}}{4s\sqrt{\pi}} \right) = \infty \tag{18}$$

which implies that (6) holds. Therefore, by Theorem (3.1) every solution of (16) is oscillatory.

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