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Majority Independence Number of a Graph

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Abstract

In this article majority independent set is introduced and majority independence number $\beta_M(G)$ is defined with examples. We discuss the relation between the majority domination number, independent majority domination number and majority independence number of a graph G.

Keywords: Majority independence number, Independent majority domination number.

I. Introduction.

By a graph, we mean a finite, simple graph which is undirected, nontrivial, without isolates. We follow the notation and terminology given by Haynes et. al [3]. Let $G=(V,E)$ be a graph of order p and size q. For every vertex $v \in V(G)$, the open neighborhood $N(v)=\{u \in V(G) / uv \in E(G)\}$ and the closed neighborhood $N[v]=N(v)\bigcup\{v\}$. Let S be a set of vertices, and let $u \in S$. The private neighbor set of u with respect to S is $pn[u,S]=\{v / N[v]\bigcap S=\{u\}\}$.

A set S of vertices in a graph $G=(V,E)$ is a dominating set if every vertex v not in S is adjacent to at least one vertex in S. A dominating set S is called a minimal dominating set if no proper subset of S is a dominating set. The domination number $\gamma(G)$ of a graph G is the minimum cardinality of a minimal dominating set in G. The upper domination number $\Gamma(G)$ of a graph G the maximum cardinality of a minimal dominating set in G. A set D of vertices in a graph G is called an independent set if no two vertices in D are adjacent. An independent set D is called a maximal independent set if any vertex set properly containing D is not independent. The independence number $\beta_o(G)$ is the maximum cardinality of a maximal independent set in G. The lower independence number $i(G)$ is the minimum cardinality of a maximal independent set of G. It is also called independent domination number of G.

Definition 1.1 [6]:
A subset $S \subseteq V(G)$ of vertices in a graph G is called majority dominating set if at least half of the vertices of $V(G)$ are either in S or adjacent to the vertices of S. i.e., $|N[S]| \geq \left\lceil \frac{p}{2} \right\rceil$. A majority dominating set S is minimal if no proper subset of S is a majority dominating set of G. The majority domination number $\gamma_M(G)$ of a graph G is the minimum cardinality of a minimal majority dominating set in G. The upper majority domination number $\Gamma_M(G)$ is the maximum cardinality of a minimal majority dominating set of a graph G. This parameter has been studied by Swaminathan V and Joseline Manora J.

II . Majority Independent Set.
Definition 2.1: A set S of vertices of a graph G is said to be a majority independent set (or MI-set) if it induces a totally disconnected subgraph with $|N[S]| \geq \left\lceil \frac{p}{2} \right\rceil$ and $|pn[v,S]| > |N[S]| - \left\lceil \frac{p}{2} \right\rceil$ for every $v \in S$. If any vertex set S' properly containing S is not majority independent then S is called maximal majority independent set. The minimum cardinality of a maximal majority independent set is called lower majority independence number of G and it is also called independent majority domination number of G. It is denoted by $i_M(G)$. The maximum cardinality of a maximal majority independent set of G is called majority independence number of G and it is denoted by $\beta_M(G)$. A β_M-set is a maximum cardinality of a maximal majority independent set of G.

Observations 2.2:
1. Every independent set of G need not be a majority independent set of G.
2. By definition, any majority independent set is a majority dominating set of a graph G.

Remarks 2.3:
1. There may be maximal independent sets which are not majority independent sets of a graph G.

Example 2.3.1: Let $G = C_{10}$. $S = (v_1, v_4, v_7, v_9)$ is a maximal independent set but it is not a majority independent set for G, since every vertex of S satisfies the condition

$|pn[v,S]|<|N[S]|-\left\lceil \frac{p}{2} \right\rceil$.

2. A maximal majority independent set need not be a maximal independent set of a graph G.

Example 2.3.2: Let $G=P_{11}$. $S_1=\{v_2,v_5,v_8\}$ is not a majority independent set since $|pn\ [v,S_1]|\leq|N[S_1]|-\left\lceil \frac{p}{2} \right\rceil$, for $\forall v\in S_1$. $S_2=\{v_2,v_5\}$ is a maximal majority independent set but S_2 is not a maximal independent set of G.

3. The relationship between $\beta_o(G)$ and $\beta_M(G)$ is $\beta_M(G)\leq\beta_o(G)$, for any graph G.

Example 2.3.3: Let $G=D_{5,4}$ be a double star with $p=11$. Suppose $D=\{v_1,v_2,....v_9\}$, for each $v\in D$, $|pn[v,D]|=1$ and $|N[D]|-\left\lceil \frac{p}{2} \right\rceil=5$. D is a β_o-set of G but D is not a β_M-set. Hence, β_o-set is not a β_M-set of G. Suppose $S=\{v_1,v_2,...v_5\}$. Then for each $v\in S$, $|pn[1,S]|=1$ and $|N[S]|-\left\lceil \frac{p}{2} \right\rceil=0$. $\therefore$ $|pn[v,S]|>|N[S]|-\left\lceil \frac{p}{2} \right\rceil$ $\forall\, v\in S$. $\therefore$ S is a majority independent set of G. Claim: S is maximal. Let $S^1=S\cup\{v_6\}$. $|N[S^1]|-\left\lceil \frac{p}{2} \right\rceil=2$ but $|pn[v,S^1]|=1 or\ 2\leq|N[S^1]|-\left\lceil \frac{p}{2} \right\rceil$ $\forall\, v\in S^1$. $\therefore$ S^1 is not a majority independent set of G. Hence S is a maximal majority independent set of G. Hence $\beta_M(G)<\beta_o(G)$.

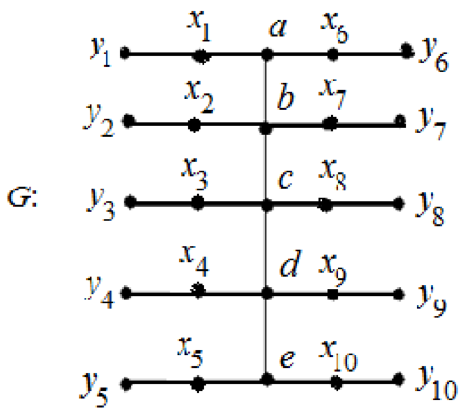
Observations 2.4:

1. For any graph G, $i_M(G)\leq\beta_M(G)$.

2. If G has a full degree vertex then that vertex constitutes a majority independent set of G.

Examples 2.4.1:

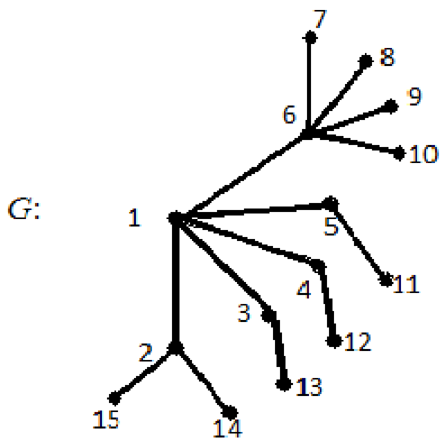
1. In G = $K_{1,6}$, Here $\beta_M(G)=3$ and $i_M(G)=1$ $\therefore i_M(G)<\beta_M(G)$
2. Let $G=K_p$. Then $i_M(G)=\beta_M(G)=1$.
3. Let $G=T_{5k}$ and k =5.



In $G = T_{25}$, $\gamma_M(G) = 4$, $i_M(G) = 4$, $\beta_M(G) = 7$ and $\beta_o(G) = 13$.

Proposition 2.5: For any graph G, $\gamma_M(G) \le i_M(G)$.

Example 2.5.1:



$\gamma_M(G) = 2$, $i_M(G) = 3$.

Theorem 2.6: Every graph G has a majority independent set.

Proof: By induction on order of G. If p=1 then $G=K_1$ and clearly G has a majority independent (MI) set namely $V(G)$. If $p=2$, then $G=K_2$ or $\overline{K_2}$. Let $S=\{u_1\}$, $N[S]=\frac{p}{2}$. $\left|pn\left[u_1,S\right]\right| > |N[S]| - \left\lceil \frac{p}{2} \right\rceil$. Therefore S is a MI-set of G. Induction hypothesis: Suppose for every graph G of order $(p-1)$, $p \ge 2$, then G has a majority independent set. Let G be a graph of order p. Let $v \in V(G)$. $G-\{v\}$ is of order $(p-1)$ $\therefore G-\{v\}$ has a majority independent set say S. i.e., $\left|N[S]\right| \geq \left\lceil \frac{p-1}{2} \right\rceil$ and $\left|pn[v,S]\right| > \left|N[S]\right| - \left\lceil \frac{p-1}{2} \right\rceil$ ……….(1)

Case (i): p is even. Now (1) becomes $\left|N[S]\right| \geq \left\lceil \frac{p-1}{2} \right\rceil = \frac{p}{2} = \left\lceil \frac{p}{2} \right\rceil$.

$\left|pn[v,S]\right| > \left|N[S]\right| - \left\lceil \frac{p-1}{2} \right\rceil = \left|N[S]\right| - \left\lceil \frac{p}{2} \right\rceil$. $\therefore$ S is a majority independent set of G.

Case (ii): p is odd then $(p-1)$ is even. By (1) $\left|N_{G-\{v\}}[S]\right| \geq \left\lceil \frac{p-1}{2} \right\rceil$ and $\left|pn_{G-\{v\}}[u,S]\right| > \left|N_{G-\{v\}}[S]\right| - \left\lceil \frac{p-1}{2} \right\rceil$ $\forall\ u \in S$. $\therefore$ $\left|N_G[S]\right| \geq \left\lceil \frac{p-1}{2} \right\rceil$ and $\left|pn_G[u,S]\right| > \left|N_G[S]\right| - \left\lceil \frac{p}{2} \right\rceil$ $\forall\ u \in S$.

Subcase (i): Suppose v is adjacent to S. Then $\left|N_G[S]\right| \geq \frac{p-1}{2} + 1 = \frac{p+1}{2} = \left\lceil \frac{p}{2} \right\rceil$.

Also, $\left|pn_G[u,S]\right| > \left|N_G[S]\right| - \left\lceil \frac{p}{2} \right\rceil$, $\forall\ u \in S$. $\therefore$ S is a majority independent set of G.

Subcase (ii): Suppose v is independent of S.

Sub subcase (i): $\left|N_{G-\{v\}}[S]\right| > \frac{p-1}{2}$. Then $\left|N_G[S]\right| = \left|N_{G-\{v\}}[S]\right| > \frac{p-1}{2}$.

$\therefore \left|N_G[S]\right| \geq \frac{p+1}{2} = \left\lceil \frac{p}{2} \right\rceil$. $\left|pn_G[u,S]\right| > \left|N_{G-\{v\}}[S]\right| - \left\lceil \frac{p-1}{2} \right\rceil > \left|N_G[S]\right| - \left\lceil \frac{p}{2} \right\rceil$ $\forall\ u \in S$. $\therefore$ S is a majority independent set of G.

Sub subcase (ii): $\left|N_{G-\{v\}}[S]\right| = \frac{p-1}{2}$. Let $S_1 = S \cup \{v\}$. S_1 is an independent set of G.

$\left|N_G[S_1]\right| = \left|N_{G-\{v\}}[S]\right| + 1 = \left\lceil \frac{p-1}{2} \right\rceil + 1 = \frac{p+1}{2} = \left\lceil \frac{p}{2} \right\rceil$. $\left|N_G[S_1]\right| - \left\lceil \frac{p}{2} \right\rceil = 0$.

Since $\left|pn_G[u,S_1]\right| = 1$ $\therefore \left|pn_G[u,S_1]\right| > \left|N_G[S_1]\right| - \left\lceil \frac{p}{2} \right\rceil$ $\forall\ u \in S_1$. $\therefore$ S_1 is a majority independent set of G. By induction, the proof is completed.

III . $\beta_M(G)$ for some standard graphs:

Computed values of $\beta_M(G)$ for some special classes graphs are stated without proof.

1. Let $G = P_p$, $p \geq 2$. Then $\beta_M(G) = \begin{cases} \left\lceil \frac{p}{4} \right\rceil & if \;\; p \leq 6. \\ \left\lceil \frac{p-2}{4} \right\rceil & if \;\; p > 6. \end{cases}$
2. Let $G = K_{1,p-1}$. Then $\beta_M(G) = \left\lfloor \frac{p-1}{2} \right\rfloor$.
3. Let $G = F_p$. Then $\beta_M(G) = \left\lceil \frac{p}{6} \right\rceil$.
4. Let $G = W_p$. Then $\beta_M(G) = \left\lceil \frac{p-2}{6} \right\rceil$.
5. Let $G = D_{r,s}$. Then $\beta_M(G) = \begin{cases} r & if \;\; r = s,\; p = r+s+2. \\ \left\lceil \frac{p}{2} \right\rceil - 1 & if \;\; r < s. \end{cases}$
6. Let $G = mK_2$. Then $\beta_M(G) = \left\lceil \frac{p}{4} \right\rceil$.
7. Let $G = \overline{K_p}$. Then $\beta_M(G) = \left\lceil \frac{p}{2} \right\rceil$.
8. Let $G = H \circ K_1$, where H is a connected graph. Then $\beta_M(G) = \left\lceil \frac{p}{4} \right\rceil$.
9. Let $G = S(K_{1,p-1})$. Then $\beta_M(G) = \left\lceil \frac{p}{4} \right\rceil$.
10. Let $G = K_p$. Then $\beta_M(G) = 1$.

Proposition 3.1: Let G be a cycle of p vertices, $p \geq 3$. Then $\beta_M(G) = \left\lceil \frac{p}{6} \right\rceil$.

Proof: Let $\{v_1, v_2, \ldots\ldots\ldots\ldots, v_p\}$ be a set of vertices of G. $d(v_i) = 2, \forall v_i \in V(G)$. Let $S = \{v_1, v_2, \ldots\ldots\ldots, v_{\beta_M(G)}\}$ be a β_M-set of G. Then $|N[S]| \geq \left\lceil \frac{p}{2} \right\rceil$ and $|pn[v,S]| > |N[S]| - \left\lceil \frac{p}{2} \right\rceil$ for every $v \in S$. Then $|N[S]| \leq \sum_{i=1}^{\beta_M(G)} d(v_i) + \beta_M(G)$. $\left\lceil \frac{p}{2} \right\rceil \leq 3\beta_M(G)$. Then $\beta_M(G) \geq \left\lceil \frac{p}{6} \right\rceil$.

Let $D = \{u_1, u_2, \ldots\ldots\ldots\ldots, u_{\left\lceil \frac{p}{6} \right\rceil + 1}\}$ with $N[u_i] \cap N[u_j] = \Phi$, $\forall\; i \neq j$. $\therefore\;\; |N[D]| = (d(u_i)+1)\left(\left\lceil \frac{p}{6} \right\rceil + 1\right) = 3\left\lceil \frac{p}{6} \right\rceil + 3 > \left\lceil \frac{p}{2} \right\rceil$. Then for $\forall u \in D$,

$|pn[u,D]| \leq |N[D]| - \left\lceil \frac{p}{2} \right\rceil \Rightarrow$ D is not a majority independent set of G. $\therefore \beta_M(G) \leq |D| = \left\lceil \frac{p}{6} \right\rceil + 1 = \left\lceil \frac{p}{6} \right\rceil$. Hence $\beta_M(G) = \left\lceil \frac{p}{6} \right\rceil$.

Proposition 3.2: Let $G = K_{m,n}$; $m \leq n$. Then

$$\beta_M(G) = \begin{cases} 1 & \text{if } m = n,\ n+1,\ n+2. \\ \left\lceil \frac{p}{2} \right\rceil - m & \text{if } m < n. \end{cases}$$

Proof: Let G be a complete bipartite graph with $V_1(G) = \{v_1, v_2, ...v_m\}$ and $V_2(G) = \{v_1, v_2, ...v_n\}$. Such that $(m+n) = p$.

Case (i): When $n = m, m+1, m+2$. Since all vertices are of degree $d(u) \geq \left\lceil \frac{p}{2} \right\rceil - 1$, $D = \{u\}$ is a maximal majority independent set of G. Hence $\beta_M(K_{m,n}) = 1$.

Case (ii): When $m < n$ and $n \geq m+3$. Let $S = \left(v_1, v_2 ....v_{\left\lceil \frac{p}{2} \right\rceil - m} \right) \subseteq V_2(G)$.

Then $|N[S]| = d(u) + |S|$ for any $u \in V_2(G)$

$= |N[u]| + |S|$, $N[u] \subseteq V_1(G)$

$= m + |S|$, where $|S| = \left\lceil \frac{p}{2} \right\rceil - m$

$= m + \left\lceil \frac{p}{2} \right\rceil - m$

$|N[S]| = \left\lceil \frac{p}{2} \right\rceil$. Then S is an independent set of G.

Next, since $|pn[v,S]| = |N[S]| - |N[S - \{v\}]|$, $|N[S - \{v\}]| = |N[S]| - |pn[v,S]|$

$\therefore |N[S]| - |pn[v,S]| < \left\lceil \frac{p}{2} \right\rceil \Rightarrow |pn[v,S]| > |N[S]| - \left\lceil \frac{p}{2} \right\rceil$. By definition S is a majority independent set of G.

Claim: S is maximal. Let $S^1 = S \cup \{u\}$, $u \in V - S$ and $S \subseteq V_2(G)$.

Subcase (i): Let $u \in V - S \subseteq V_2(G)$. $|N[S^1]| = |N[S]| + |N[u]| = \left\lceil \frac{p}{2} \right\rceil + 1 \geq \left\lceil \frac{p}{2} \right\rceil$.

Next, since $|pn[u, S^1]| = |N[S^1]| - |N[S^1 - \{u\}]|$

$$\left|pn\left[u,S^1\right]\right|=\left\lceil\frac{p}{2}\right\rceil+1-\left\lceil\frac{p}{2}\right\rceil=1$$

$$\left|N[S^1]\right|-\left\lceil\frac{p}{2}\right\rceil=\left\lceil\frac{p}{2}\right\rceil+1-\left\lceil\frac{p}{2}\right\rceil=1=\left|pn[u,S^1]\right|$$

$\therefore$ $\left|pn\left[u,S^1\right]\right|=\left|N[S^1]\right|-\left\lceil\frac{p}{2}\right\rceil$, $\forall\, u\in V_2(G)$. $\therefore$ S^1 is not a majority independent set of G.

Subcase (ii): Let $u\in V-S\subseteq V_1(G)$. Let $S^1=S\bigcup\{u\}$ $\Rightarrow S^1$ is not an independent set of G. $\therefore$ S^1 is not a majority independent set of G. Hence, S is a maximal majority independent set of G. $\therefore$ $\beta_M(G)\geq|S|=\left\lceil\frac{p}{2}\right\rceil-m$.

Conversely, Suppose $S=\left\{u_1,u_2,......u_{\left\lceil\frac{p}{2}\right\rceil}\right\}\subseteq V_2(G)$. Then S is an independent set and $|S|=\left\lceil\frac{p}{2}\right\rceil$. Now, $|N[S]|=\left\lceil\frac{p}{2}\right\rceil+m\geq\left\lceil\frac{p}{2}\right\rceil$. For $\forall u\in S$,

$$\left|pn[u,S]\right|=\left|N[S]\right|-\left|N\left[S-\{u\}\right]\right|$$

$$=\left(\left\lceil\frac{p}{2}\right\rceil+m\right)-\left(m+|S|-1\right)=\left\lceil\frac{p}{2}\right\rceil-|S|+1=1.$$

Next, $|N[S]|-\left\lceil\frac{p}{2}\right\rceil=\left\lceil\frac{p}{2}\right\rceil+m-\left\lceil\frac{p}{2}\right\rceil=m,\ m\geq1$. $\therefore\left|pn\left[u,S\right]\right|\leq|N[S]|-\left\lceil\frac{p}{2}\right\rceil$, $\forall u\in S\subseteq V_2(G)$. $\therefore$ S is not a majority independent set of G. Since $S\subseteq V_2(G)$, any vertex in $V_1(G)$ is not an element in S. Otherwise, S is not an independent set of G. Since $|N[S]|=\left\lceil\frac{p}{2}\right\rceil+m$, in order to decrease the order of $|S|$, now take $|S|=\left\lceil\frac{p}{2}\right\rceil-m$. Then, $|N[S]|=\left\lceil\frac{p}{2}\right\rceil$ and for $\forall\ u\in S$, $|pn[u,S]|>|N[S]|-\left\lceil\frac{p}{2}\right\rceil$. $\therefore$ $\beta_M(G)\leq\left\lceil\frac{p}{2}\right\rceil-m$. Combining these we get, $\beta_M(G)=\left\lceil\frac{p}{2}\right\rceil-m$, if $G=K_{m,n},\ m<n$.

IV. Some results on majority independent sets.

Theorem 4.1: For any graph G, $\beta_M(G)=1$ if and only if G has all vertices with $d(u)\geq\left\lceil\frac{p}{2}\right\rceil-1$ for $\forall u\in G$.

Theorem 4.2: A majority independent set S of a graph G is maximal majority independent if and only if it is majority independent and majority dominating set of

G.

Proof: Let S be a maximal majority independent set of G. Then S is majority independent set and $|N[S]| \geq \left\lceil \frac{p}{2} \right\rceil \Longrightarrow$ S is a majority dominating set of G. Conversely, S is both majority independent and majority dominating set. Then $|N[S]| \geq \left\lceil \frac{p}{2} \right\rceil$ and $|pn\ [v, S]| > |\ N[S]\ | - \left\lceil \frac{p}{2} \right\rceil$ for every $v \in S$. Then there $\exists u \in V - S$ such that S ∪ {u} = S^1 with $|N[S']| > \left\lceil \frac{p}{2} \right\rceil$ then $|pn\ [v, S']| \leq |\ N[S']\ | - \left\lceil \frac{p}{2} \right\rceil\ \forall u \in S'$. It implies that S^1 is not a majority independent set. Hence S is a maximal majority independent set of G.

Theorem 4.3: Every maximal majority independent set of G is a minimal majority dominating set of G.

Proof: Let S be a maximal majority independent set of G. Then S is a majority dominating set. Suppose S is not minimal, there exists at least one $v \in S$ such that S-{v} is a majority dominating set of G. $\Rightarrow |N[S - \{v\}]| \geq \left\lceil \frac{p}{2} \right\rceil$ Since S is maximal majority independent set of G, $|N[S]| \geq \left\lceil \frac{p}{2} \right\rceil$ and $pn\ [v, S] > |\ N[S]\ | - \left\lceil \frac{p}{2} \right\rceil$............(1), for $\forall\ v \in S$, $and\ |pn[v, S]| \geq 1$. We know that, $pn[v, S] = N[S] - N[S - \{v\}]$. ⇒ $|N[S - \{v\}]| = |N[S]| - pn[v,S]$ < $|N[S]| - |N[S]| + \left\lceil \frac{p}{2} \right\rceil$ by (1). $\therefore |N[S - \{v\}]| < \left\lceil \frac{p}{2} \right\rceil$ $\Rightarrow (S - \{v\})$ is not a majority dominating set, a contradiction. Hence S is minimal majority dominating set of G.

Proposition 4.4: For any graph G, γ_M(G) ≤ β_M (G).

Proof: Let S be a β_M - set of G. By theorem 4.3, then S is a minimal majority dominating set of G. Therefore $\gamma_M(G) \leq |S| = \beta_M(G)$.

Proposition 4.5: For any graph G, γ_M (G) ≤ $i_M(G)$ ≤ β_M (G).

Proof: From Proposition 2.5 and Theorem 4.2.

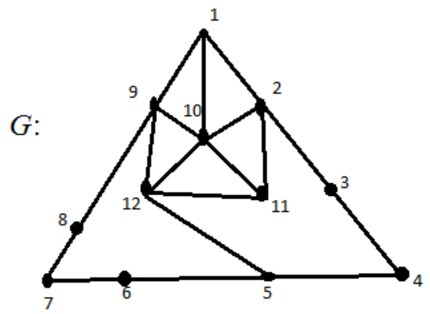
Example 4.5.1:

1. $\gamma_M(G) = i_M(G) = \beta_M(G) = 1 \; if \; G = K_p$.
2. For $G = C_{10}$, $\gamma_M(G) = i_M(G) = \beta_M(G) = 2$.

Proposition 4.6: For any graph G, $\gamma_M(G) \le i_M(G) \le \beta_M(G) \le \Gamma_M(G)$.

Proof: Since every minimal majority dominating set with maximum cardinality of G is a majority dominating set of G, we get this result.

Example 4.6.1:



$\gamma_M(G) = i_M(G) = 1, \quad \beta_M(G) = 2, \quad \Gamma_M(G) = 3$.

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