International Journal of Mathematics Research.
ISSN 0976-5840 Volume 6, Number 1 (2014), pp. 93-97

http://www.irphouse.com

Generalisation on the Zeros of a Family of Complex Polynomials

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Abstract

In this paper we prove some results on the location of zeros of a certain class of polynomials. These results generalize some known results in the theory of the distribution of zeros of polynomials. Hence our result will considerably improve the bounds by relaxing and weakening the hypothesis in some cases. Here we obtain certain generalizations and refinements of well known Enestrom – Kakeya Theorem for a polynomial under much less restrictions on its coefficients.

Keywords: Polynomials, Zeros, Enestrom-Kakeya theorem, Analytic functions

INTRODUCTION

Many results on the location of zeros of polynomials are available in the literature. Among them the Enestrom-Kakeya theorem [4] given below is well known in the theory of zero distribution of polynomials.

Theorem (i). For an nth-order polynomial P(z) = $\sum_{i=0}^{n} a_i z^i$, assume

$$a_n \geq a_{n-1} \geq a_{n-2} \geq\text{------}\geq a_1 \geq a_0 > 0 \qquad 1$$

Then P(z) has all its zeros in the disk $|z| \leq 1$.

In the literature [1-18], diverse attempts have been made for generalizing the Enestrom-Kakeya theorem to polynomials and analytic functions.

Recently, Choo[5] also proved the following theorems:

Theorem (ii). Consider an nth-order complex polynomial P(z) $=\sum_{i=0}^{n} a_i z^i$, with Re(a_i) = α_i and Im(a_i) = β_i, i = 0, 1, 2-----n, and assume that for some k and r, and for λ_1, λ_2 and t > 0,

$$\lambda_1 t^n \alpha_n \le t^{n-1}\alpha_{n-1} \le \ldots\ldots\ldots \le t^{k+1}\alpha_{k+1} \le t^k \alpha_k \ge \ldots\ldots \ge t\alpha_1 \ge \alpha_0,$$
$$\lambda_2 t^n \beta_n \le t^{n-1}\beta_{n-1} \le \ldots\ldots\ldots \le t^{r+1}\beta_{r+1} \le t^r \beta_r \ge \ldots\ldots \ge t\beta_1 \ge \beta_0. \tag{2}$$

Then P (z) has all its zeros in $R_1 \le |z| \le R_2$ where

$$R_1 = \frac{t|a_0|}{M_1} \; and \;\; R_2 = \frac{M_2}{t^{n-1}|a_n|}, \tag{3}$$

with

$$M_1 = t^n|a_n| + t^n|(\lambda_1 - 1)\alpha_n| + t^n|(\lambda_2 - 1)\beta_n| + 2(t^k\alpha_k + t^r\beta_r) - t^n(\lambda_1\alpha_n + \lambda_2\beta_n) - (\alpha_0 + \beta_0), \tag{4(i)}$$

and

$$M_2 = t^n|(\lambda_1 - 1)\alpha_n| + t^n|(\lambda_2 - 1)\beta_n| + 2(t^k\alpha_k + t^r\beta_r) - t^n(\lambda_1\alpha_n + \lambda_2\beta_n) - (\alpha_0 + \beta_0) + |a_0|. \tag{4(ii)}$$

Theorem (iii). Consider an nth-order complex polynomial P(z) $=\sum_{i=0}^{n} a_i z^i$, with Re(a_i) = α_i and Im(a_i) = β_i, i = 0, 1, 2-----n, and assume that for some k, λ and for some t > 0,

$$\lambda\, t^n\alpha_n \le t^{n-1}\alpha_{n-1} \le \ldots\ldots\ldots \le t^{k+1}\alpha_{k+1} \le t^k\alpha_k \ge \ldots\ldots \ge t\alpha_1 \ge \alpha_0, \tag{5}$$

Then P(z) has all its zeros in $R_1 \le |z| \le R_2$ where

$$R_1 = \frac{t|a_0|}{M_1} \; and \;\; R_2 = \frac{M_2}{t^{n-1}|a_n|}, \tag{6}$$

with

$$M_1 = t^n|a_n| + t^n|(\lambda - 1)\alpha_n| + 2t^k\alpha_k - t^n\lambda\,\alpha_n - \alpha_0 + t^n|\beta_n| + |\beta_0| + 2\sum_{j=1}^{n-1} t^j|\beta_j|, \tag{6(i)}$$

and

$$M_2 = t^n|(\lambda - 1)\alpha_n| + 2t^k\alpha_k - t^n\lambda\alpha_n - \alpha_0 + |a_0| + t^n|\beta_n| + |\beta_0| + 2\sum_{j=1}^{n-1} t^j|\beta_j|,. \tag{6(ii)}$$

Now we prove the following theorem:-

Theorem 1. Consider an nth-order complex polynomial $P(z) = \sum_{i=0}^{n} a_i z^i$, with $Re(a_i) = \alpha_i$ and $Im(a_i) = \beta_i$, i = 0, 1, 2-----n, and assume that for some p and q, and for $\delta, \eta > 0$ and $0 \leq \tau, \sigma \leq 1$,

$$\begin{aligned}&\delta t^n \alpha_n \leq t^{n-1}\alpha_{n-1} \leq ....\leq t^{p+1}\alpha_{p+1} \leq t^p \alpha_p \geq ....\geq t\alpha_1 \geq \tau\alpha_0,\\ &\eta t^n \beta_n \leq t^{n-1}\beta_{n-1} \leq ....\leq t^{r+1}\beta_{q+1} \leq t^q \beta_q \geq ......\geq t\beta_1 \geq \sigma\beta_0.\end{aligned} \tag{7}$$

Then P (z) has all its zeros in $R_1 \leq |z| \leq R_2$ where

$$R_1 = \frac{t|a_0|}{M_1} \text{ and } R_2 = \frac{M_2}{t^{n-1}|a_n|}, \tag{8}$$

with

$$\begin{aligned}M_1 &= t^n|a_n| + t^n|(\delta-1)\alpha_n| + t^n|(\eta-1)\beta_n| + 2(t^p\alpha_p + t^q\beta_q) - t^n(\delta\alpha_n + \eta\beta_n) - \tau\alpha_0\\ &+ (1-\tau)\alpha_0 - \sigma\beta_0 + (1-\sigma)\beta_0\end{aligned} \tag{8(i)}$$

and

$$\begin{aligned}M_2 &= t^n|(\delta-1)\alpha_n| + t^n|(\eta-1)\beta_n| + 2(t^p\alpha_p + t^q\beta_q) - t^n(\delta\alpha_n + \eta\beta_n) -\\ &(\tau\alpha_0 + \sigma\beta_0) + |1-\tau|\alpha_0 + |1-\sigma|\beta_0 + |a_0|.\end{aligned} \tag{8(ii)}$$

Proof: Firstly we consider the case where t = 1. For the outer bound, consider a polynomial

$$\begin{aligned}G(z) &= (1-z)P(z)\\ &= -\alpha_n z^{n+1} + \{(\alpha_n - \delta\alpha_n) + (\delta\alpha_n - \alpha_{n-1})\}z^n + (\alpha_{n-1} - \alpha_{n-2})z^{n-1} + ---- + (\alpha_{p+1} - \alpha_p)z^{p+1} + (\alpha_p - \alpha_{p-1})z^p +\\ &(\alpha_{p-1} - \alpha_{p-2})z^{p-1} + ----- + \{(\alpha_1 - \tau\alpha_0) + (\tau\alpha_0 - \alpha_0)\}z + \alpha_0 + i[-\beta_n z^{n+1} + \{(\beta_n - \eta\beta_n) + (\eta\beta_n - \beta_{n-1})\}z^n +\\ &(\beta_{n-1} - \beta_{n-2})z^{n-1} + ---- + (\beta_{q+1} - \beta_q)z^{q+1} + (\beta_q - \beta_{q-1})z^q + (\beta_{q-1} - \beta_{q-2})z^{q-1}\\ &+ -------------- + \{(\beta_1 - \sigma\beta_0)\\ &(\sigma\beta_0 - \beta_0)\}z + \beta_0]\end{aligned} \tag{9}$$

Now if $|z|>1$, $\frac{1}{|z|^{n-j}} < 1$, j= 0, 1, 2---n-1,

$$|G(z)| \geq |z|^n \{|a_n||z|^{-}M_{12}\} \tag{10}$$

Where $M_{12} = |(\delta-1)\alpha_n| + |(\eta-1)\beta_n| - (\delta\alpha_n + \eta\beta_n) + 2(\alpha_p + \beta q) - (\tau\alpha_0 + \sigma\beta_0) + (1-\tau)\alpha_0 +$

$$(1-\sigma)\beta_0 + |a_0| \tag{11}$$

Then $|G(z)| \geq 0$ if $|z| > \frac{M_{12}}{|a_n|} = R_{12}$ and all the zeros of P(z) with modulus greater then one lie in the disk $|z| \leq R_{12}$. It can be shown that $R_{12} \geq 1$. Consequently the zeros of P(z) with modulus less than or equal to one are already contained in the disk $|z| \leq R_{12}$.

For the inner bounds, again consider $G(z) = H(z) + a_0$ (12)

If $|z| < 1$ then
Therefore $|H(z)| \leq M_{11}$ (13)

Where
$M_{11} = |a_n| + |(\delta-1)\alpha_n| + |(\eta-1)\beta_n| - (\delta\alpha_n + \eta\beta_n) + 2(\alpha_p + \beta q) - (\tau\alpha_0 + \sigma\beta 0) + (1-\tau)\alpha_0 + (1-\sigma)\beta 0$ (14)

Since $H(0) = 0$, it follows that Schwarz lemma that
$|H(z)| \leq M_{11}|z|$ for $|z| < 1$
Then $|z| < 1$, $G(z) \geq |a_0| - |H(z)| \geq |a_0| - M_{11}|z| > 0$ (15)

If $|z| < \frac{|a_0|}{M_{11}} = R_{11}$ then it can be shown that $R_{11} \leq 1$. Hence if $t = 1$ then all the zeros of $P(z)$ lie in the disk $R_{11} \leq |z| \leq R_{12}$. (16)

It is now easy to find the result of the above theorem follows from the result applicable to $P(tz)$. Hence the proof of the above theorem is complete.

Corollary:-If in the above theorem we substitute for each of the above parameters τ and σ equal to unity then the above results coincides with results obtained by Choo[5].

Conclusion:-
Here in our theorem we showed the refinement over Choo[5].

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