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Best Approximation and Fixed Points for Nonconvex Sets

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ABSTRACT

The notion of T-regular sets was introduced in vector spaces by P. Veeramani [J. Math. Anal. Appl. 167 (1992), 160-166] to generalize a well known fixed point theorem of F. E. Browder: If T is a nonexpensive self map on a weakly compact convex subset of a uniformly convex Banach space then T has a fixed point. We extend this notion of T-regular sets to convex metric spaces and use it to prove results on the invariance of best approximation in such spaces thereby generalizing some of the earlier known results in metrizable topological vector spaces.
The notion of T-regular sets was introduced in vector spaces by Veeramani [8] to generalize a well known fixed point theorem of Browder(Corollary 1. 3[8]) and to obtain a result on invariant approximation (Corollary 2. 2[8]) in Banach spaces. This class of nonconvex sets was extensively used by Khan and Hussain [3] to study iterative approximation of fixed points of non-expansive mappings. Khan, Bano and Hussain [4] also obtained best approximation results using fixed points and obtained Corollary 2. 2 of Veeramani [8] without the assumptions of uniform convexity of the space and non-expansiveness of the mapping in the setting of metrizable topological vector spaces(Theorem 2. 9[4]). We extend this notion of T-regular sets to convex metric spaces and generalize some results of Khan and Hussain [3] and of Khan, Bano and Hussain [4] proved in metrizable topological vector spaces to convex metric spaces.
To start with, we recall a few definitions.
Let M be a non-empty subset of a metric space (X, d). For any $x\in X$, the set $P_M(x) = \{y\in M: d(x, y) = d(x, M)\}$, where $d(x, M) = \inf\{ d(x, z): z\in M \}$ is called the set of best approximants to x in M.

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The set M is said to be approximatively compact if for every $x \in X$ and every sequence $<x_n>$ in M with $\lim d(x, x_n) = \text{dist}(x, M)$, there exists a subsequence $< x_{n_i} >$ converging to an element of M.

For a metric space (X, d) and I = [0, 1], a continuous mapping $W: X \times X \times I \to X$ is said to be Takahashi convex structure (TCS) on X if for all $x, y, u \in X$ and $t \in [0, 1]$,

$d(u, W(x, y, t)) \leq t\, d(u, x) + (1-t)\, d(u, y)$.

A metric space (X, d) with a Takahashi convex structure is called a convex metric space [6].

A normed linear space and each of its convex subsets are simple examples of convex metric spaces. There are many convex metric spaces which are not normed linear spaces (see [6]).

A convex metric space (X, d) is said to be strict TCS [7] if it has the property that whenever $w \in X$ and there is $(x, y, t) \in X \times X \times I$ for which $d(z, w) \leq t\, d(z, x) + (1-t)\, d(z, y)$, for every $z \in X$ then $w = W(x, y, t)$.

A convex metric space (X, d) is said to satisfy Property (III) [2], if

$d(W(x, y, t), W(u, v, t)) \leq t\, d(x, u) + (1-t)\, d(y, v)$.

for $x, y, u, v \in X$ and $t \in [0, 1]$.

A convex metric space (X, d) is said to be strictly convex [1] if for all $x, y, z \in X$ and $r > 0$, $d(x, z) \leq r$, $d(y, z) \leq r$ imply $d(W(x, y, 1/2), z) < r$ unless $x = y$.

A convex metric space (X, d) is said to be uniformly convex [1] if there corresponds to each pair of positive numbers (ε, r) a positive number δ such that for all $x, y, z \in X$, $d(x, z) \leq r$, $d(y, z) \leq r$ and $d(x, y) \geq \varepsilon$ imply $d(W(x, y, 1/2), z) \leq r-\delta$.

Clearly every uniformly convex metric space is strictly convex but converse is not true[1].

A subset M of a convex metric space (X, d) is said to be convex[6] if $W(x, y, t) \in M$ whenever $x, y \in M$ and $t \in I$, and it is said to be T-regular if

(i) $T: M \to M$,

(ii) $W(x, Tx, 1/2) \in M$ for each $x \in M$.

The following results are easy to prove:

(a) Every convex set invariant under a map T is a T-regular set but a T-regular set need not be convex.
(b) Union and intersection of T-regular sets are T-regular.
(c) If $T: X \to X$ is a continuous mapping and M is T-regular then $cl(M) \equiv$ closure of M is T-regular.

Let $M = \{a, b, W(a, b, 1/2)\}$, $a \neq b$ be a subset of a strict TCS space (X, d). Define $T: M \to M$ as $T(a) = b$, $T(b) = a$, $T(c) = c$, where $c = W(a, b, 1/2)$. Since $W(c, Tc, 1/2) = W(c, c, 1/2) = c$ and in a strict TCS space $W(a, b, 1/2) = W(b, a, 1/2)$ (see[7]-

propositions 1. 2 and 2. 4), the set M is compact and T-regular but not convex.

Define the bisection map F: M→M as Fx = W(x, Tx, 1/2). As W(a, Fa, 1/2) =W(a, W(a, Ta, 1/2), 1/2) = W(a, W(a, b, 1/2), 1/2) $\notin$ M, M is not F-regular.

If T is non-expansive map on a T-regular set M of a convex metric space (X, d) then the bisection map F may not be non-expansive. However, we have

Proposition1. If (X, d) is a convex metric space with Property (III) and T is non-expansive on a T-regular set M then the bisection map F is non-expansive on M..

Proof. Consider
$d(Fx, Fy) = d(W(x, Tx, 1/2), W(y, Ty, 1/2))$
$\leq ½ d(x, y) + ½ d(Tx, Ty)$
$\leq ½ d(x, y) + ½ d(x, y)$
$= d(x, y)$
for all x, y $\in$ M.

For a subset M of a metric space (X, d), let $D(M) = \sup\{d(x, y): x, y \in M\} \equiv D$ be the diameter of M. A point $x_o \in M$ is called a diametral point of M if $\sup\{d(x_o, y): y \in M \}= D(M)$.

For T-regular sets we have:

Proposition2. Let (X, d) be a uniformly convex metric space and M a bounded T-regular subset of X. Then either each point of M is a fixed point of T or there exists a nondiametral point u of M i. e. a point $u \in M$ such that $\sup\{d(u, y): y \in M \}\equiv\delta(u, M)< D(M)$.

Proof. Suppose there exists some $x \in M$ such that $Tx\neq x$. Consider u = W(x, Tx, 1/2). Then $u\in M$. We claim that u is the desired point.

Let $y\in M$ be arbitrary. Then $d(y, Tx) \leq D$ and $d(y, x) \leq D$. Let $\varepsilon = d(x, Tx) > 0$. By the uniform convexity of the space there exists a $\delta > 0$ such that $d(y, W(x. Tx, 1/2)) \leq D-\delta$ i. e. $d(y, u) \leq D-\delta$ and so $\sup \{ d(u, y): y\in M \} \leq D-\delta < D$ i. e. $u \in M$ is non-diametral point.

Note1. For metrizable uniformly convex topological vector spaces, Proposition 2 was proved in [3] and for uniformly Convex Banach spaces it was proved in [8].

For strictly convex metric spaces, we have

Proposition3. Let (X, d) be strictly convex metric space, $x \in X$ and M a subset of X. If $y_1 \neq y_2 \in P_M(x)$ then $W(y_1, y_2, t) \notin M$, $0 < t < 1$.

Proof. $y_1, y_2 \in P_M(x) \Rightarrow d(x, y_1) = d(x, M) = d(x, y_2)$. Since (X, d) is strictly convex, $d(x, W(y_1, y_2, t)) < d(x, M)$ and so $W(y_1, y_2, t) \notin M$.

Note 2. For strictly convex metric linear spaces, Proposition 3 was proved in [5].

Using Proposition 3, we prove the following result on invariant approximation:

Proposition 4. Let (X, d) be strictly convex metric space, M any subset of X and $T:M \rightarrow M$. If $P_M(x)$ is non-empty and T-regular for any $x \in X$, then each point of $P_M(x)$ is a fixed point of T.

Proof. Suppose for some $u \in P_M(x)$, $u \neq Tu$. Then by Proposition 3, $W(u, Tu, 1/2) \notin M$ and so it cannot be in $P_M(x)$. Since $P_M(x)$ is T-regular, we must have u=Tu i. e. each best approximation of x is a fixed point of T.

Note 3. For strictly convex metrizable topological vector spaces, Proposition 4 was proved in [3].

The following theorem on invariant approximation generalizes and extends Theorem 2. 9 [4] to strictly convex metric spaces:

Theorem1. Let M be a non empty T-regular subset of strictly convex metric space (X, d) and u a point of X. Suppose that $d(Tx, u) \leq d(x, u)$ for all $x \in M$. Then each x in M which is a best approximation to u, is a fixed point of T provided one of the following conditions hold:
M is closed and T is a compact mapping i. e. T(M) is contained in a compact subset of M.
M is proximinal.
M is approximatively compact.

Proof. Suppose (i) holds. Let $r = d(u, M)$. Then there is a sequence $\langle y_n \rangle$ in M such that $\lim d(u, y_n) = r$. This sequence $\langle y_n \rangle$ is a bounded sequence. As T is compact, $cl(\{Ty_n\})$ is a compact subset of M and so $\langle Ty_n \rangle$ has a convergent subsequence $\langle Ty_{n_i} \rangle \rightarrow x \in M$. Consider

$$r \leq d(u, x) = \lim d(u, Ty_{n_i}) \leq \lim d(u, y_{n_i}) = r$$

and so $x \in P_M(u)$. Also, if $y \in P_M(u)$ then $Ty \in M$ and $r \leq d(Ty, u) \leq d(y, u) = r$ imply

$Ty \in P_M(u)$, so $d(y, u) = r = d(Ty, u)$. Therefore, by the convexity of X, $d(u, W(y, Ty, 1/2)) \leq r$

and hence $W(y, Ty, 1/2) \in P_M(u)$ as $W(y, Ty, 1/2) \in M$ i. e. $P_M(u)$ is T-regular. The result now follows from Proposition 4.

Suppose (ii) holds. Since M is proximinal, $P_M(u)$ is non-empty and so result follows as in (i).

Suppose (iii) holds. Since M is approximatively compact, M is proximinal (see [1]) and the proofs follows from (ii).

By applying Theorem1, we obtain another result on invariant approximation in strictly convex metric spaces.

Theorem2. Let (X, d) be a strictly convex metric space and $T: X \rightarrow X$ be a mapping. If M is a non-empty T-regular subset of X, $u \in X \setminus M$ is such that Tu=u and T is a generalized non-expansive map i. e.

$d(Tx, Ty) \leq \alpha\, d(x, y) + \beta\, [d(x, Tx)+d(y, Ty)] + \gamma\, [d(x, Ty)+d(y, Tx)]$
for all $x, y \in X$, where α, β, γ are real numbers with $\alpha+2\beta+2\gamma \leq 1$. Then each $x \in P_M(u)$ is a fixed point of T provided one of the conditions (i)-(iii) of Theorem 1 hold:

Proof. Since
$d(Tx, Tu) \leq \alpha\, d(x, u) + \beta\, [d(x, Tx) + d(u, Tu)] + \gamma\, [d(x, Tu) + d(u, Tx)]$
$\leq \alpha\, d(x, u) + \beta\, d(x, Tx) + \gamma\, [d(x, u) + d(Tu, Tx)]$
$\leq \alpha\, d(x, u) + \beta\, [d(x, u) + d(u, Tx)] + \gamma\, [d(x, u) + d(Tu, Tx)]$

This gives

$d(Tx, Tu) \leq \{(\alpha+\beta+\gamma) / (1-\alpha-\beta)\} d(x, u).$

Therefore $d(Tx, Tu) \leq d(x, u)$ for all $x \in X$. The result now follows from Theorem1.

Note 4. For strictly convex metrizable topological vector spaces, Theorem 2 was proved in [4]. It may be remarked that Theorem 2 is valid even if T is only a quasi non-expansive mapping i. e. it satisfies $d(Tx, Tu) \leq d(x, u)$ for all $x \in X$ and $u \in F(T)$.

The following example justifies Proposition 4, Theorems1 and 2.

Example[8]. Let $M = [-2,-1] \cup [1, 2]$. Define $T: M \to M$ as

$$T(x) = \begin{cases} -1, & x \in [-2,-1] \\ 1, & x \in [1,2] \end{cases}$$

Then all the conditions of the above mentioned results are satisfied. $P_M(o) = \{-1, 1\}$ and each point of $P_M(o)$ is a fixed point of T.

Using properties of T-regular sets, we prove the following theorem:

Theorem 3: Let K be a non empty compact T-regular subset of a uniformly convex metric space (X, d) and for each closed T-regular subset F of K with $D(F) > 0$ there exists some $\beta(F)$, $0 < \beta(F) < 1$ such that

$d(Tx, Ty) \leq \max. \{d(x, y), \beta\, D(F)\}$ for all $x, y \in F$.

Then T has a fixed point in K.

Proof: Let $\aleph$ be the collection of all non-empty closed T-regular subsets of K. In view of fact that union and intersection of T-regular sets are T-regular, one can use Zorn's Lemma to get a minimal element, say F, of $\aleph$.

Suppose for some x in F, $x \neq Tx$. Since F is a bounded T-regular set, Proposition 2 implies that there exists $x_0 \in F$ and α, $0 < \alpha < 1$ such that

$\delta(x_0, F) \leq \alpha\, D(F)$

Also, by hypothesis, there exists β, $0 < \beta < 1$ such that $d(Tx, Ty) \leq \beta\, D(F)$.

Let $\alpha_0 = \max\{\alpha, \beta\}$, $F_0 = \{z \in F: \delta(z, F) \leq \alpha_0\, D(F)\}$. Then F_0 is non empty as $x_0 \in F$ and F_0 is a closed set. Let $z \in F_0$ then $d(Tz, Ty) \leq d(z, y) \leq \alpha_0\, D(F)$ for all $y \in F$. Therefore $T(F) \subset U \equiv B\alpha_{0\, D(F)}\,[Tx]$, a closed ball with centre Tx and radius $\alpha_0\, D(F)$.

This gives $T(F \cap U) \subset F \cap U$. Since F is T-regular and U is a convex set, $F \cap U$ is T-regular and so is in the collection $\aleph$. Hence by the minimality of F, $Tx \in F_0$ i. e. $T(F_0) \subset F_0$. Also F_0 is a T-regular set and so $F_0 \in \aleph$. But $D(F_0) < D(F)$ if F contains more than one point, a contradiction. Therefore, F contains exactly one element and this is invariant under T. Hence Tx = x

Corollary 1. Let K be a non empty compact T-regular subset of uniformly convex metric space (X, d) and T: $K \to K$ be a non-expensive mapping then T has a fixed point.

Corollary 2. Let K be a non empty compact convex subset of uniformly convex metric space (X, d) and T: $K \to K$ be a non-expensive mapping then T has a fixed point.

Remark: Theorem 3 for weakly compact T-regular subsets of uniformly convex Banach spaces was proved by Veeramani [8] and the above proof is a minor modification of the one given in [8].

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