International Journal of Mathematics Research.
ISSN 0976-5840 Volume 6, Number 2 (2014), pp. 135-142
© International Research Publication House
http://www.irphouse.com

On Partial Quasi-bilateral Generating Functions Involving Gegenbauer Polynomials

K.P. Samanta[1] and B. Samanta[2]

[1]Department of Mathematics, Indian Institute of Engineering Science and Technology, Shibpur, Howrah–711103, W.B., India.
[2]Department of Mathematics, Shibpur D.B. Institution(College), 412/1, G.T. Road, Howrah–711102, W.B., India
E-mail: 1kalipadasamanta2010@gmail.com, [2]bijoysamanta@yahoo.com

Abstract

In this paper, we have obtained a generalization of a known result on quasi-bilateral generating relation involving Gegenbauer polynomials from the existence of partial quasi-bilateral generating relation of the polynomial under consideration. Some particular cases of interest are also pointed out.

AMS-2010 Subject Classification Code: 33C45, 33C47, 33C65

Keywords:Gegenbauer polynomials, partial quasi-bilateral generating functions.

1. Introduction

In [1],Mondal defined partial quasi-bilateral generating for two special functions by means of the relation:

$$G(x, u, w) = \sum_{n=0}^{\infty} a_n\, P_{m+n}^{(\alpha)}(x)\, Q_r^{(m+n)}(u) w^n,$$

where a_n, the coefficients are quite arbitrary and $P_{m+n}^{(\alpha)}(x)$, $Q_r^{(m+n)}(u)$ are two particular special functions of orders $m+n$, r and of parameters α and $m+n$ respectively. If $Q_r^{(m+n)}(u) \equiv P_r^{(m+n)}(u)$, the generating relation is known as partial quasi-bilinear.

In this note, we would like to show that the existence of a partial quasi-bilinear generating function implies the existence of a more general generating function from the group theoretic view-point.

In [2], Samanta, Chandra and Bera have proved the following theorem on bilateral generating functions involving modified Gegenbauer poynomials, $C_n^{\lambda+n}(x)$ by group-theoretic method.

Theorem 1 If there exists a unilateral generating relation of the form

$$G(x, w) = \sum_{n=0}^{\infty} a_n C_n^{\lambda+n}(x)\, w^n \qquad (1.1)$$

then

$$\frac{(1-w)^{\lambda-\frac{1}{2}}}{\{1-w+wx^2\}^{\lambda}} G\left(\frac{x}{\{1-w+wx^2\}^{\frac{1}{2}}}, \frac{wv(1-w)}{\{1-w+wx^2\}^{\frac{3}{2}}}\right) \qquad (1.1)$$

$$= \sum_{n=0}^{\infty} w^n \sigma_n(x, v), \qquad (1.2.)$$

where

$$\sigma_n(x, v) = \sum_{p=0}^{n} a_p \frac{\left(\frac{p+1}{2}\right)_{n-p}\left(\frac{p+2}{2}\right)_{n-p}}{(n-p)!(1-\lambda-p)_{n-p}} C_{2n-p}^{\lambda-n+2p}(x)\, v^p.$$

The importance of the above theorem lies in the fact that whenever one knows a unilateral generating relation of the form (1.1) then the corresponding bilateral generating relation can at once be written down from (1.2). So one can get a large number of bilateral generating relations by attributing different suitable values to a_n in (1.1).

Subsequently, In [3], Samanta and Chongdar obtained an extension of the above theorem in the following form:

Theorem 2 If there exists a unilateral generating relation of the form

$$G(x, w) = \sum_{n=0}^{\infty} a_n C_{n+r}^{\lambda+n}(x)\, w^n \qquad (1.3)$$

Then

$$\frac{(1-w)^{\lambda-\frac{1}{2}}}{\{1-w+wx^2\}^{\lambda+\frac{r}{2}}} G\left(\frac{x}{\{1-w+wx^2\}^{\frac{1}{2}}}, \frac{wv(1-w)}{\{1-w+wx^2\}^{\frac{3}{2}}}\right) = \sum_{n=0}^{\infty} w^n \sigma_n(x, v), \qquad (1.4)$$

Where

$$\sigma_n(x, v) = \sum_{p=0}^{n} a_p \frac{\left(\frac{p+r+1}{2}\right)_{n-p}\left(\frac{p+r+2}{2}\right)_{n-p}}{(n-p)!\,(1-\lambda-p)_{n-p}} C_{2n+r-p}^{\lambda-n+2p}(x)\, v^p.$$

In [4], authors have obtained a nice extension of the Theorem 1 from the existence of quasi-bilinear generating relation.

Theorem 3 If there exists a quasi-bilinear generating relation of the following form

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n C_n^{\lambda+n}(x) C_m^{n}(u) w^n$$

then

$$(1-2w)^{\lambda-\frac{m}{2}-\frac{1}{2}}\{1-2w(1-x^2)\}^{-\lambda}$$

$$\times G\left(\frac{x}{\{1-2w(1-x^2)\}^{\frac{1}{2}}}, \frac{u}{(1-2w)^{\frac{1}{2}}} \frac{wt}{\{1-2w(1-x^2)\}^{\frac{3}{2}}}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} a_n \frac{w^{n+p+q}}{p!q!} 2^{p+q} \frac{\left(\frac{n+1}{2}\right)_p \left(\frac{n+2}{2}\right)_p (n)_q}{(1-\lambda-n)_p} C_{n+2p}^{\lambda+n-p}(x) C_m^{n+q}(u) t^n.$$

The object of the present paper is to further generalize the above theorem from the concept of partial quasi-bilateral(or partial quasi-bilinear) generating functions. In fact, we have obtained the following theorem as the main result of our investigation.

Theorem 4 If there exists a partial quasi-bilinear generating relation of the following form

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n C_{n+r}^{\lambda+n}(x) C_m^{n+r}(u) w^n$$

Then

$$(1-2w)^{\lambda-\frac{m}{2}-\frac{1}{2}-r}\{1-2w(1-x^2)\}^{-\frac{r}{2}-\lambda}$$

$$\times G\left(\frac{x}{\{1-2w(1-x^2)\}^{\frac{1}{2}}}, \frac{u}{(1-2w)^{\frac{1}{2}}} \frac{wt}{\{1-2w(1-x^2)\}^{\frac{3}{2}}}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} a_n \frac{w^{n+p+q}}{p!q!} 2^{p+q} \frac{\left(\frac{n+r+1}{2}\right)_p \left(\frac{n+r+2}{2}\right)_p (n+r)_q}{(1-\lambda-n)_p} C_{n+r+2p}^{\lambda+n-p}(x) C_m^{n+r+q}(u) t^n.$$

2. Proof of the theorem

At first we consider the generating relation of the form:

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n C_{n+r}^{\lambda+n}(x) C_m^{n+r}(u) w^n. \tag{2.1}$$

For the Gegenbauer polynomials, we consider the following operators[3,5]:

$$R_1 = x(1-x^2)\frac{y^2}{z^3}\frac{\partial}{\partial x} + (1-3x^2)\frac{y^3}{z^3}\frac{\partial}{\partial y} - \frac{2x^2y^2}{z^2}\frac{\partial}{\partial z} + (1+r-rx^2)\frac{y^2}{z^3},$$

$$R_2 = uv\frac{\partial}{\partial u} + 2v^2\frac{\partial}{\partial v} + (m+2r)v$$

such that

$$R_1\left(C_{n+r}^{\lambda+n}(x)y^n z^\lambda\right) = \frac{(n+r+1)(n+r+2)}{2(1-\lambda-n)} C_{n+r+2}^{\lambda+n-1}(x)y^{n+2}z^{\lambda-3}, \tag{2.2}$$

$$R_2(C_m^{n+r}(u)v^n) = 2(n+r)C_m^{n+r+1}(\mathrm{u})v^{n+1} \tag{2.3}$$

And

$$e^{wR_1}f(x,\ y,\ z) = \left\{1-2w\frac{y^2}{z^3}\right\}^{-\frac{1}{2}}\left\{1-2w(1-x^2)\frac{y^2}{z^3}\right\}^{-\frac{r}{2}}$$

$$\times f\left(\frac{x}{\left\{1-2w(1-x^2)\frac{y^2}{z^3}\right\}^{\frac{1}{2}}},\ \frac{y\left(1-2w\frac{y^2}{z^3}\right)}{\left\{1-2w(1-x^2)\frac{y^2}{z^3}\right\}^{\frac{3}{2}}},\ \frac{z\left(1-2w\frac{y^2}{z^3}\right)}{\left\{1-2w(1-x^2)\frac{y^2}{z^3}\right\}}\right), \tag{2.4}$$

$$e^{wR_2}f(u,v) = (1-2wv)^{-\frac{m}{2}-r}f\left(\frac{u}{(1-2wv)^{\frac{1}{2}}},\frac{v}{(1-2wv)}\right) \tag{2.5}$$

Replacing w by $wvyt$ and multiplying both sides of (2.1) by z^λ, we get

$$z^\lambda\ G(x,\ u, wvyt) = \sum_{n=0}^{\infty} a_n\left(C_{n+r}^{\lambda+n}(x)y^n z^\lambda\right)\left(C_m^{n+r}(u)v^n\right)(wt)^n. \tag{2.6}$$

Now operating $e^{wR_1}\ e^{wR_2}$ on both sides of (2.6), we get

$$e^{wR_1}e^{wR_2}\left[z^\lambda G(x,\ u, wvyt)\right] =$$

$$e^{wR_1}e^{wR_2}\left[\textstyle\sum_{n=0}^{\infty} a_n\left(C_{n+r}^{\lambda+n}(x)y^n z^\lambda\right)\left(C_m^{n+r}(u)v^n\right)(wt)^n\right] \tag{2.7}$$

The left member of (2.7), with the help of (2.4) and (2.5), becomes

$$\left(1-2w\frac{y^2}{z^3}\right)^{\lambda-\frac{1}{2}}\left\{1-2w(1-x^2)\frac{y^2}{z^3}\right\}^{-\frac{r}{2}-\lambda}(1-2wv)^{-\frac{m}{2}-r}z^\lambda$$

$$\times G\left(\frac{x}{\left\{1-2w(1-x^2)\frac{y^2}{z^3}\right\}^{\frac{1}{2}}},\frac{u}{(1-2wv)^{\frac{1}{2}}}\frac{wvyt\left(1-2w\frac{y^2}{z^3}\right)}{\left\{1-2w(1-x^2)\frac{y^2}{z^3}\right\}^{\frac{3}{2}}(1-2wv)}\right) \tag{2.8}$$

The right member of (2.7), with the help of (2.2) and (2.3), becomes

$$\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} a_n \frac{w^{p+q}}{p!\,q!} 2^{p+q} \frac{\left(\frac{n+r+1}{2}\right)_p \left(\frac{n+r+2}{2}\right)_p (n+r)_q}{(1-\lambda-n)_p}$$

$$\times\, C_{n+r+2p}^{\lambda+n-p}(x) C_m^{n+r+q}(u) y^{n+2p} z^{\lambda-3p} v^{n+q} (wt)^n. \qquad (2.9)$$

Now equating both members, and then substituting $y = z = v = 1$, we get

$$(1-2w)^{\lambda-\frac{m}{2}-\frac{1}{2}-r}\left\{1-2w(1-x^2)\right\}^{-\frac{r}{2}-\lambda}$$

$$\times\, G\left(\frac{x}{\{1-2w(1-x^2)\}^{\frac{1}{2}}}, \frac{u}{(1-2w)^{\frac{1}{2}}}, \frac{wt}{\{1-2w(1-x^2)\}^{\frac{3}{2}}}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} a_n \frac{w^{n+p+q}}{p!q!} 2^{p+q} \frac{\left(\frac{n+r+1}{2}\right)_p \left(\frac{n+r+2}{2}\right)_p (n+r)_q}{(1-\lambda-n)_p} C_{n+r+2p}^{\lambda+n-p}(x) C_m^{n+r+q}(u) t^n. \qquad (2.10)$$

This completes the proof of Theorem 4.

Corollary 1: Putting $r = 0$ in (10), we get

$$(1-2w)^{\lambda-\frac{m}{2}-\frac{1}{2}}\left\{1-2w(1-x^2)\right\}^{-\lambda}$$

$$\times\, G\left(\frac{x}{\{1-2w(1-x^2)\}^{\frac{1}{2}}}, \frac{u}{(1-2w)^{\frac{1}{2}}}, \frac{wt}{\{1-2w(1-x^2)\}^{\frac{3}{2}}}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} a_n \frac{w^{n+p+q}}{p!q!} 2^{p+q} \frac{\left(\frac{n+1}{2}\right)_p \left(\frac{n+2}{2}\right)_p (n)_q}{(1-\lambda-n)_p} C_{n+2p}^{\lambda+n-p}(x) C_m^{n+q}(u) t^n,$$

whichis Theorem 3. Thus Theorem 4 is an extension of Theorem 3.

Corollary 2: If we put $m = 0$, we notice that $G(x, u, w)$ becomes $G(x, w)$ since $C_0^{n+r+q}(u) = 1$. Hence from (2.10), we get

$$(1-2w)^{\lambda-\frac{1}{2}-r}\left\{1-2w(1-x^2)\right\}^{-\frac{r}{2}-\lambda}$$

$$\times\, G\left(\frac{x}{\{1-2w(1-x^2)\}^{\frac{1}{2}}}, \frac{wt}{\{1-2w(1-x^2)\}^{\frac{3}{2}}}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty} a_n \frac{w^{n+p}}{p!} 2^p \frac{\left(\frac{n+r+1}{2}\right)_p \left(\frac{n+r+2}{2}\right)_p}{(1-\lambda-n)_p} C_{n+r+2p}^{\lambda+n-p}(x) t^n \left(\sum_{q=0}^{\infty} \frac{(2w)^q (n+r)_q}{q!}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty} a_n \frac{(2w)^{n+p}}{p!} \frac{\left(\frac{n+r+1}{2}\right)_p \left(\frac{n+r+2}{2}\right)_p}{(1-\lambda-n)_p} C_{n+r+2p}^{\lambda+n-p}(x) \left\{\frac{t}{2(1-2w)}\right\}^n (1-2w)^{-r}$$

Replacing $\left(\frac{t}{2(1-2w)}\right)$ by v and then $2w$ by w on both sides, we get

$$(1-w)^{\lambda-\frac{1}{2}}\{1-w(1-x^2)\}^{-\frac{r}{2}-\lambda} G\left(\frac{x}{\{1-w(1-x^2)\}^{\frac{1}{2}}}, \frac{wv(1-w)}{\{1-w(1-x^2)\}^{\frac{3}{2}}}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty} a_n \frac{(w)^{n+p}}{p!} \frac{\left(\frac{n+r+1}{2}\right)_p \left(\frac{n+r+2}{2}\right)_p}{(1-\lambda-n)_p} C_{n+r+2p}^{\lambda+n-p}(x) v^n$$

$$= \sum_{n=0}^{\infty}\sum_{p=0}^{n} a_{n-p} \frac{w^n}{p!} \frac{\left(\frac{n-p+r+1}{2}\right)_p \left(\frac{n-p+r+2}{2}\right)_p}{(1-\lambda-n+p)_p} C_{n+r+p}^{\lambda+n-2p}(x) v^{n-p}.$$

Therefore we have

$$(1-w)^{\lambda-\frac{1}{2}}\{1-w(1-x^2)\}^{-\frac{r}{2}-\lambda} G\left(\frac{x}{\{1-w(1-x^2)\}^{\frac{1}{2}}}, \frac{wv(1-w)}{\{1-w(1-x^2)\}^{\frac{3}{2}}}\right)$$

$$= \sum_{n=0}^{\infty} w^n \sigma_n(x,v),$$

Where

$$\sigma_n(x,v) = \sum_{p=0}^{n} a_p \frac{\left(\frac{p+r+1}{2}\right)_{n-p} \left(\frac{p+r+2}{2}\right)_{n-p}}{(n-p)!(1-\lambda-p)_{n-p}} C_{2n+r-p}^{\lambda-n+2p}(x)\, v^p.$$

Corollary 3: If we put $r = 0$ in the above result, we get the Theorem 1.

3. Acknowledgment

The authors are greatly indebted to Dr. A.K. Chongdar of IIEST, Shibpur for his constant encouragement and guidance.

References

[1] Mondal, A. K. : A unified group-theoretic method of obtaining more general class of generating relations from a given class of partial quasi-bilateral generating relations involving some special functions, pure Math. Manuscript, 10 (1992-93), 133-140.

[2] Samanta, K.P., Chandra B.C. and Bera C.S., On generating functions of Gegenbauer polynomials, Ultra Scientist Vol. 26(1)A (2014), 74-78.

[3] Samanta,K.P. and Chongdar, A.K.:On some generating functions of modified Gegenbauer polynomials by group-theoretic method, Jour. of Sci. Arts, 1(26) (2014), 27-38.

[4] Samanta, K.P., Chandra B.C. and Bera C.S. :An extension of bilateral generating functions of Gegenbauer polynomials, J. of Ramanujan Society of Math. and Math. Sc., 2(2) (2014), 71-76.

[5] Mondal, A. K. : Partial quasi-bilateral generating function involving Hermite and Gegenbauer polynomials, pure Math. Manuscript, 10 (1992-93), 81-87.