International Journal of Mathematics Research.
ISSN 0976-5840 Volume 6, Number 2 (2014), pp. 149-156

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Generalized Mixed Equilibrium Problems with Extended Relaxed α-monotonicity

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Abstract

In this paper, we introduce the notion of extended relaxed α-monotonicity. We obtain existence solution of the generalized mixed equilibrium problems in reflexive Banach spaces with extended relaxed α-monotonicity by using the KKM technique. Our results extend and improve the corresponding results in the existing literature.

Keywords: Equilibrium problem, generalized mixed equilibrium problem, extended relaxed α-monotonicity, reflexive Banach space, convexity.

INTRODUCTION

Let K be a nonempty subset of a real reflexive Banach space X. Let Let $\varphi : K \to \mathbb{R}$ be a real-valued function and $f : K \times K \to \mathbb{R}$ be an equilibrium bi-function, i.e., $f(x, x) = 0, \forall\, x \in K$. Then the mixed equilibrium problem (for short, MEP) is to find $\bar{x} \in K$ such that

$$f(\bar{x}, y) + \varphi(y) - \varphi(\bar{x}) \geq 0, \forall\, y \in K$$

In particular, if $\varphi \equiv 0$, this problem reduces to the classical equilibrium problem (for short, EP) which is to find $\bar{x} \in K$ such that

$$f(\bar{x}, y) \geq 0\ , \forall\, y \in K$$

Equilibrium problems and mixed equilibrium problems play an important role in many fields, such as economics, physics, mechanics and engineering sciences. Also, the equilibrium problems and mixed equilibrium problems include many mathematical problems as particular cases for example, mathematical programming problems, complementary problems, variational inequality problems, Nash equilibrium problems in noncooperative games, minimax inequality problems, and fixed point problems. Because of their wide applicability, equilibrium problems and mixed equilibrium problems have been generalized in various directions for the past several years.

The monotonicity and generalized monotonicity play an important role in the study of equilibrium problems and mixed equilibrium problems. In recent years, a substantial number of papers on existence results for solving equilibrium problems and mixed equilibrium problems based on different generalization of monotonicity such as pseudomonotonicity, quasimonotonicity, relaxed monotonicity, semimonotonicity, see, [1-17].

In 1990, Karamardian and Schaible [14] introduced various kinds of generalized monotone mappings. Afterward, several researcher in [2-4] extended the idea of Karamardian and Schaible [14] for bi-functions to study equilibrium problems.

In 2003, Fang and Huang [10] considered two classes of the variational-like inequalities with the relaxed η-α monotone and relaxed η-α semimonotone mappings. They obtained the existence solutions of variational-like inequalities with relaxed η-α monotone and relaxed η-α semimonotone mappings in Banach spaces using the KKM technique. Later Bai et al.[1] introduced a new concept of relaxed η-α pseudomonotone mappings and obtained the solutions for the variational-like inequalities. Afterward Mahato and Nahak [18] defined the weakly relaxed η-α pseudomonotone bi-function to study the equilibrium problems.

Recently Mahato and Nahak [19] introduce the concept of the relaxed α-monotonicity for bi-functions. They also obtained the existence of solutions for mixed equilibrium problems with the relaxed α-monotone bi-function in reflexive Banach spaces, by using the KKM technique.

Inspired and motivated by the recent research developed for mixed equilibrium problems and variational inequalities, the purpose of this paper is to introduce the class of extended relaxed α-monotone bi-functions. The existence of solutions for generalized mixed equilibrium problems with bi-function in such class is given. Our results in this paper extend and improve the results of Mahato and Nahak [19] and many results in the literature.

PRELIMINARIES

Let K be a nonempty closed convex subset of a real reflexive Banach space X. Let $\varphi : K \to \mathbb{R}$ be a real-valued function and $f : K \times K \to \mathbb{R}$ be a bi-function such that $f(\lambda x + (1-\lambda)z, x) = 0,\ \forall\, x, z \in K$ and $\lambda \in (0,1]$. Then the generalized mixed equilibrium problems (for short, GMEP) is to find $\bar{x} \in K$ such that

$$f(\lambda\bar{x} + (1-\lambda)z, y) + \varphi(y) - \varphi(\bar{x}) \geq 0, \forall\, y, z \in K \text{ and } \lambda \in (0,1]. \quad (2.1)$$

In particular, if $\lambda = 1, \varphi \equiv 0$, this problem reduces to the classical equilibrium problem (for short, EP), which is to find $\bar{x} \in K$ such that

$$f(\bar{x}, y) \geq 0, \forall\, y \in K \quad (2.2)$$

The motivation of studying the generalized mixed equilibrium problems is to cover the mixed variational inequality problems, which has been studied in [10,20]. Throughout the paper, unless otherwise stated, K be a nonempty closed convex subset of a reflexive Banach space X. We consider the mappings $f : K \times K \to \mathbb{R}$ and $\alpha : X \times X \to \mathbb{R}$.

Definition 2.1 A bi-function $f : K \times K \to \mathbb{R}$ is said to be extended relaxed α-monotone if

$$f(\lambda x + (1-\lambda)z, y) + f(\lambda y + (1-\lambda)z, x) \leq \alpha(y, x), \forall\, x, y, z \in K \text{ and } \lambda \in (0,1] \quad (2.3)$$

where $\lim_{t\to 0} \frac{\alpha(ty+(1-t)x,x)}{t} = 0$

Remark 2.1 (i) If $\lambda = 1, \alpha(y, x) = \beta(y - x)$, where $\beta : X \to \mathbb{R}$ with $\beta(tu) = t^p\beta(u)$, for $t > 0$ and $u \in X$; then the definition 2.1 reduces to relaxed α-monotone mapping, i.e.

$$f(x, y) + f(y, x) \leq \beta(y - x), \forall x, y \in K \quad (2.4)$$

(ii) If $\lambda = 1$, $\alpha \equiv 0$ then from definition 2.1, it follows that f is monotone, i.e.,

$$f(x, y) + f(y, x) \leq 0, \forall x, y \in K \quad (2.5)$$

Definition 2.2 A real valued function f defined on a convex subset K of X is said to be hemicontinuous if $\lim_{t\to 0^+} f(tx + (1-t)y) = f(y)$, for each $x, y \in K$.

Definition 2.3 Let $f : K \to 2^X$ be a set-valued mapping. Then f is said to be KKM mapping if for any $\{y_1, y_2, ..., y_n\}$ of K we have $co\{y_1, y_2, ..., y_n\} \subset \cup_{i=1}^{n} f(y_i)$, where $co\{y_1, y_2, ..., y_n\}$ denotes the convex hull of $\{y_1, y_2, ..., y_n\}$.

Lemma 2.1 [21] Let M be a nonempty subset of a Hausdroff topological vector space X and let $f : M \to 2^X$ be a KKM mapping. If $f(y)$ is closed in X for all $y \in M$ and compact for some $y \in M$, then

$$\bigcap_{y\in M} f(y) \neq \emptyset$$

Definition 2.4 Let X be a Banach space. A function $f : X \to \mathbb{R}$ is lower semicontinuous at $x_0 \in X$ if

$$f(x_0) \leq \liminf_n f(x_n)$$

for any sequence $\{x_n\} \in X$ such that x_n converges to x_0.

Definition 2.5 Let X be a Banach space. A function $f : X \to \mathbb{R}$ is weakly upper semicontinuous at $x_0 \in X$ if

$$f(x_0) \geq \limsup_n f(x_n)$$

for any sequence $\{x_n\} \in X$ such that x_n converges to x_0 weakly.

EXISTENCE RESULTS

Now we discuss the existence solution of the generalized mixed equilibrium problem (2.1), using the concept of the extended relaxed α-monotonicity of the bi-function f. These existence results generalize the corresponding results of variational-like inequality problems [10] to equilibrium problems.

Theorem 3.1 Suppose $f : K \times K \to \mathbb{R}$ with $f(\lambda x + (1-\lambda)z, x) = 0$, $\forall x, z \in K$, $\lambda \in (0,1]$ be extended relaxed α-monotone which is hemicontinuous in the first argument, and convex in the second argument ; let $\varphi : K \to \mathbb{R}$ be a convex function. Then, the (GMEP) and the following problem (3.1) are equivalent:

Find $\bar{x} \in K$ such that

$$f(\lambda y + (1-\lambda)z, \bar{x}) + \varphi(\bar{x}) - \varphi(y) \le \alpha(y, \bar{x}), \forall y, z \in K\ , \lambda \in (0,1]. \tag{3.1}$$

Proof Suppose that GMEP (2.1) has a solution, i.e., there exists $\bar{x} \in K$ such that

$$f(\lambda\bar{x} + (1-\lambda)z, y) + \varphi(y) - \varphi(\bar{x}) \ge 0, \forall\, y, z \in K, \lambda \in (0,1].$$

Since f is extended relaxed α-monotone, we have

$$\begin{aligned} &f(\lambda y + (1-\lambda)z, \bar{x}) + \varphi(\bar{x}) - \varphi(y) \le \alpha(y, \bar{x}) - f(\lambda\bar{x} + (1-\lambda)z, y) + \varphi(\bar{x}) - \varphi(y) \\ &= \alpha(y, \bar{x}) - [f(\lambda\bar{x} + (1-\lambda)z, y) + \varphi(y) - \varphi(\bar{x})] \\ &\le \alpha(y, \bar{x}), \forall\, y, z \in K \end{aligned} \tag{3.2}$$

Hence $\bar{x} \in K$ is a solution of (3.1).

Conversely, suppose $\bar{x} \in K$ is a solution of (3.1) and $y \in K$ be any point. Letting $x_t = ty + (1-t)\,\bar{x}$, $t \in (0,1]$, as K is convex, $x_t \in K$. Therefore from (3.1) we have

$$f(\lambda\, x_t + (1-\lambda)z, \bar{x}) + \varphi(\bar{x}) - \varphi(\,x_t) \le \alpha(\,x_t, \bar{x}), \forall z \in K, \lambda \in (0,1] \tag{3.3}$$

It follows from the convexity of f in the second argument that,

$$0 = f(\lambda x_t + (1-\lambda)z\,, x_t) \le tf(\lambda x_t + (1-\lambda)z, y) + (1-t)f(\lambda x_t + (1-\lambda)z, \bar{x})$$

$$\Rightarrow t[f(\lambda x_t + (1-\lambda)z, \bar{x}) - f(\lambda x_t + (1-\lambda)z, y)] \le f(\lambda x_t + (1-\lambda)z, \bar{x}) \tag{3.4}$$

and the convexity of φ implies that,

$$0 = \varphi(\,x_t) - \varphi(\,x_t) \le t\varphi(y) + (1-t)\varphi(\bar{x}) - \varphi(\,x_t)$$

$$\Rightarrow t[\varphi(\bar{x}) - \varphi(y)] \le \varphi(\bar{x}) - \varphi(\,x_t) \tag{3.5}$$

From (3.3), (3.4) and (3.5), we have

$$t[f(\lambda x_t + (1-\lambda)z, \bar{x}) - f(\lambda x_t + (1-\lambda)z, y) + \varphi(\bar{x}) - \varphi(y)] \le \alpha(\,x_t, \bar{x})$$

$$\Rightarrow\ f(\lambda x_t + (1-\lambda)z, \bar{x}) - f(\lambda x_t + (1-\lambda)z, y) + \varphi(\bar{x}) - \varphi(y) \le \frac{\alpha(\,x_t, \bar{x})}{t}$$

Since f is hemicontinuous in the first argument and taking $t \to 0$, we get

$$f(\lambda\bar{x} + (1-\lambda)z, \bar{x}) - f(\lambda\bar{x} + (1-\lambda)z, y) + \varphi(\bar{x}) - \varphi(y) \le 0$$

$$\Rightarrow\ f(\lambda\bar{x} + (1-\lambda)z, y) +\ \varphi(y) - \varphi(\bar{x}) \ge 0\ , \forall\ y\ , z \in K.$$

Hence $\bar{x}$ is a solution of GMEP.

Theorem 3.2 Let K be a nonempty bounded closed convex subset of a real reflexive Banach space X. Suppose $f : K \times K \to \mathbb{R}$ with $f(\lambda x + (1-\lambda)z, x) = 0\ ,\ \forall x, z \in K\ , \lambda \in (0,1]$. be extended relaxed α-monotone and hemicontinuous in the first argument ; let $\varphi : K \to \mathbb{R}$ be a convex and lower semicontinuous function. Assume that For fixed $z \in K$, the mapping $x \longmapsto f(z, x)$ is convex and lower semicontinuous; $\alpha : X \times X \to \mathbb{R}$ is weakly upper semicontinuous in the second argument.

Then the (GMEP) has a solution.

Proof Consider the set valued mapping $F : K \to 2^X$ such that $F(y) = \{x \in K : f(\lambda x + (1-\lambda)z, y) + \varphi(y) - \varphi(x) \ge 0\}\,, \forall\ y\ , z \in K, \lambda \in (0,1]$.

It is easy to see that $\bar{x} \in K$ solves the (GMEP) $f(\lambda\bar{x} + (1-\lambda)z, y) +\ \varphi(y) - \varphi(\bar{x}) \ge 0\ ,\ \forall\ y\ , z \in K, \lambda \in (0,1]$, if and only if $\bar{x} \in \bigcap_{y\in K} F(y)$. Thus it suffices to prove $\bigcap_{y\in K} F(y) \ne \emptyset$.

First we claim that F is a KKM mapping.

If F is not a KKM mapping, then there exists $\{x_1, x_2, \ldots, x_m\} \subset K$ such that $co\{x_1, x_2, \ldots, x_m\} \not\subseteq \bigcup_{i=1}^{m} F(x_i)$, that means there exists $x_0 \in\ co\{x_1, x_2, \ldots, x_m\}\,, x_0 = \sum_{i=1}^{m} t_i x_i$, where $t_i \ge 0, i = 1,2, \ldots, m, \sum_{i=1}^{m} t_i = 1$, but $x_0 \notin \bigcup_{i=1}^{m} F(x_i)$.

From the construction of F, we have

$f(\lambda x_0 + (1-\lambda)z, x_i) + \varphi(x_i) - \varphi(x_0) < 0$; for $i = 1,2,\dots,m$ (3.6)

From (3.6), and the convexity of f in second variable and φ, it follows that

$$\begin{aligned} 0 = f(\lambda x_0 + (1-\lambda)z, x_0) &= f\left(\lambda x_0 + (1-\lambda)z, \sum_{i=1}^{m} t_i x_i\right) \\ &\leq \sum_{i=1}^{m} t_i f(\lambda x_0 + (1-\lambda)z, x_i) \\ &< \sum_{i=1}^{m} t_i [\varphi(x_0) - \varphi(x_i)] \\ &= \varphi(x_0) - \sum_{i=1}^{m} t_i \varphi(x_i) \\ &\leq \varphi(x_0) - \varphi(x_0) \\ &= 0 \end{aligned}$$

which is a contradiction. Hence F is a KKM mapping.

Define another set valued mapping $G : K \to 2^X$ such that

$G(y) = \{x \in K : f(\lambda y + (1-\lambda)z, x) + \varphi(x) - \varphi(y) \leq \alpha(y, x)\}, \forall\, y, z \in K, \lambda \in (0,1]$

Now we will prove that $F(y) \subset G(y)$

For any given $y, z \in K$, $\lambda \in (0,1]$, let $x \in F(y)$ then

$$f(\lambda x + (1-\lambda)z, y) + \varphi(y) - \varphi(x) \geq 0$$

From the extended relaxed α-monotonicity of f, it follows that

$$\begin{aligned} f(\lambda y + (1-\lambda)z, x) + \varphi(x) - \varphi(y) &\leq \alpha(y, x) - [f(\lambda x + (1-\lambda)z, y) + \varphi(y) - \varphi(x)] \\ &\leq \alpha(y, x) \end{aligned}$$

Therefore $x \in G(y)$, i.e., $F(y) \subset G(y)$, $\forall\, y, z \in K, \lambda \in (0,1]$.

This implies that G is also a KKM mapping.

Since $x \longmapsto f(z, x)$ and φ are convex lower semicontinuous functions, we know that they are both weakly lower semicontinuous. From the definition of G and the weakly upper semicontinuity of α in the second argument, it is easy to see that $G(y)$ is weakly closed for all $y, z \in K$, $\lambda \in (0,1]$. Since K is closed bounded and convex, it is weakly compact, and so $G(y)$ is weakly compact in K for each $y, z \in K$, $\lambda \in (0,1]$. Therefore from Lemma 2.1 and Theorem 3.1, it follows that

$$\bigcap_{y \in K} F(y) = \bigcap_{y \in K} G(y) \neq \emptyset$$

So there exists $\bar{x} \in K$, such that $f(\lambda \bar{x} + (1-\lambda)z, y) + \varphi(y) - \varphi(\bar{x}) \geq 0$, $\forall\, y, z \in K, \lambda \in (0,1]$,i.e., (GMEP) has a solution.

Theorem 3.3 Let K be a nonempty unbounded closed convex subset of a real reflexive Banach space X. Suppose $f : K \times K \to \mathbb{R}$ with $f(\lambda x + (1-\lambda)z, x) = 0$, $\forall x, z \in K, \lambda \in (0,1]$ be extended relaxed α-monotone which is hemicontinuous in the first argument ; let $\varphi : K \to \mathbb{R}$ be a convex and lower semicontinuous function. Assume that For fixed $z \in K$, the mapping $x \longmapsto f(z, x)$ is convex and lower semicontinuous; $\alpha : X \times X \to \mathbb{R}$ is weakly upper semicontinuous in the second argument; f satisfied the weakly coercivity condition : i.e. there exists $x_0 \in K$ such that $f(\lambda x + (1-\lambda)z, x_0) + \varphi(x_0) - \varphi(x) < 0$, whenever $\|x\| \to +\infty$ and $x, z \in K$

Then the (GMEP) has a solution.

Proof For $r > 0$, assume $K_r = \{ y \in K : \|y\| \leq r\}$.

Consider the problem: find $x_r \in K \cap K_r$ such that

$$f(\lambda x_r + (1-\lambda)z, y) + \varphi(y) - \varphi(x_r) \geq 0, \forall y, z \in K \cap K_r \tag{3.7}$$

By Theorem 3.2, we know that problem (3.7) has at least one solution $x_r \in K \cap K_r$

Choose $r > \|x_0\|$ with x_0 as in the coercivity condition (iii). Then $x_0 \in K \cap K_r$, and

$$f(\lambda x_r + (1-\lambda)z, x_0) + \varphi(x_0) - \varphi(x_r) \geq 0. \tag{3.8}$$

If $\|x_r\| = r$ for all r, we may choose r large enough so that by the coercivity condition (iii), we have $f(\lambda x_r + (1-\lambda)z, x_0) + \varphi(x_0) - \varphi(x_r) < 0$, which contradicts (3.8).

Therefore there exists r such that $\|x_r\| < r$. For any $y \in K$, we can choose $0 < t < 1$ small enough such that $x_r + t(y - x_r) \in K \cap K_r$

From (3.7), it follows that

$$\begin{aligned} 0 &\leq f\big(\lambda x_r + (1-\lambda)z, x_r + t(y-x_r)\big) + \varphi\big(x_r + t(y-x_r)\big) - \varphi(x_r) \\ &\leq tf(\lambda x_r + (1-\lambda)z, y) + (1-\mathrm{t})f(\lambda x_r + (1-\lambda)z, x_r) + t\,\varphi(y) + (1-t)\varphi(x_r) - \varphi(x_r) \\ &= t[f(\lambda x_r + (1-\lambda)z, y) + \varphi(y) - \varphi(x_r)] \end{aligned}$$

Hence $f(\lambda x_r + (1-\lambda)z, y) + \varphi(y) - \varphi(x_r) \geq 0, \forall y \in K$

Therefore, the (GMEP) has a solution. This completes the proof.

CONCLUSION

The present work has been aimed to theoretically study the existence of solutions for generalized mixed equilibrium problem under extended relaxed α-monotonicity in reflexive Banach spaces. Our results generalize the mixed equilibrium problems as well as the notion of relaxed α-monotone mapping.

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