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Power Filtering of n-th Order in the Linear Canonical Domain

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Abstract

Power filtering in Linear Canonical transform is related to n-th order derivative of the Linear Canonical transform. In this paper we discuss the main properties of power filtering in Linear Canonical domain and its relation to the differentiation operation.

1. Introduction

Linear Canonical transform is the generalization of Fractional Fourier transform. The kernel of Linear Canonical transform depends on 2×2 matrix and hence Linear Canonical transform has three more parameter than Fractional Fourier transform. The Linear Canonical transform of the function $f(x)$ for any matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by

$$L^A[f(u)](x) = F_A(u) = \int_{-\infty}^{\infty} f(x) K_A(u,x)dx$$

Where $$K_A(u,x) = \sqrt{\frac{1}{2\pi ib}} exp\left(\frac{i}{2b}[du^2 - 2ux + ax^2]\right)$$

Pei and Ding [4] had studied its eigen functions and eigen valued. Gudadhe [3] discussed about its convolution and applied it for linear shift invariant system in [2]. Sharma and Joshi [5] applied it to signal seperation. Sampling in Linear Canonical transform domain is discussed by Stern [6].

The term filtering is generally used in engineering and applied mathematics which means to eliminate some frequency component entirely from the input signals. Mostly it is useful in signal processing, image processing etc. where the signals we wish to recover are degraded by distortion, blur, noise etc Fourier transform is the commonly used integral transform for filtering due to its linear time invariant property. Operation transform and its relation with the differentiation operation are widely used to solve the differential equations and also the signal derivatives are related to the power filtering in the Fourier transform and Fractional Fourier transform domain. Power filtering in Fourier transform and Fractional Fourier transform domain were widely studied in past years [1]. But power filtering in Linear Canonical transform domain not yet and this is the subject of the paper.

Fractional Fourier transform is frequently used in signal analysis, mostly in filtering. Alieva et. al. [1] considered power filtering in Fractional Fourier domain. In this paper we had discussed power filtering in Linear Canonical transform domain and investigated some of its properties.

2. Power Filtering of n-th Order in the Linear Canonical Domain-

Using the operation transform formula based on multiplication rule

$$L^{\mp A}[F_{\pm A}(u) \cdot u](x) = axf(x) \mp \frac{b}{2\pi i}\frac{d}{dx}f(x) \tag{1}$$

then n-th order power filtering operation can be written in the form

We start with the case $n = 1$, which is related to first order derivative of the signal $f(x)$. Filtering in the Linear Canonical domain with mask u yields the following result

$$L^{\mp A}[F_{\pm A}(u) \cdot u](x) = axf(x) \mp \frac{b}{2\pi i}\frac{d}{dx}f(x) \tag{2}$$

$$= aL^{A_1}\left[F_{A_1}(u) \cdot u\right](x) + \frac{b}{2\pi i}L^{A_2^{-1}}\left[F_{A_2}(u) \cdot u\right](x)$$

where $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ which can be considered as weighted sum of corresponding filtering results in the position and Fourier domain. Taking eq. for two different matrices P and Q, we can find the first derivative $\frac{df}{dx} = 2\pi i L^{A_2^{-1}}\left[F_{A_2}(u) \cdot u\right](x)$ and the product $xf(x) = L^{A_1}\left[F_{A_1}(u) \cdot u\right](x)$

For matrices $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $Q = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$

$$L^{-P}[F_P(u) \cdot u](x) = axf(x) - \frac{b}{2\pi i}\frac{d}{dx}f(x)$$

$$L^{-Q}\left[F_Q(u) \cdot u\right](x) = pxf(x) - \frac{q}{2\pi i}\frac{d}{dx}f(x)$$

Multiplying above eq. by p and a respectively

$$pL^{-P}[F_P(u)\cdot u](x) = paxf(x) - \frac{pb}{2\pi i}\frac{d}{dx}f(x)$$

$$aL^{-Q}\left[F_Q(u)\cdot u\right](x) = apxf(x) - \frac{aq}{2\pi i}\frac{d}{dx}f(x)$$

Adding them we get

$$\frac{-(pb-aq)}{2\pi i}\frac{d}{dx}f(x) = pL^{-P}[F_P(u)\cdot u](x) - aL^{-Q}\left[F_Q(u)\cdot u\right](x)$$

$$\frac{-1}{2\pi i}\frac{d}{dx}f(x) = \frac{1}{(pb-aq)}\left\{pL^{-P}[F_P(u)\cdot u](x) - aL^{-Q}\left[F_Q(u)\cdot u\right](x)\right\}$$

Similarly

$$xf(x) = \frac{-1}{(pb-aq)}\left\{qL^{-P}[F_P(u)\cdot u](x) - bL^{-Q}\left[F_Q(u)\cdot u\right](x)\right\}$$

From (1) we also derived that sum of squared moduli of the filtered signal- first-order power filtered in the A^+ and A^- Canonical transform domain is related to square moduli of the signal derivative and the signal intensity as

$$|L^{-A}[F_+(u)\cdot u](x)|^2 = a^2|xf(x)|^2 - 2ab(i2\pi)^{-1}|xf(x)|\cdot\left|\frac{d}{dx}f(x)\right| + b^2(i2\pi)^{-2}\left|\frac{d}{dx}f(x)\right|^2 \quad (3)$$

$$|L^{+A}[F_-(u)\cdot u](x)|^2 = a^2|xf(x)|^2 + 2ab(i2\pi)^{-1}|xf(x)|\cdot\left|\frac{d}{dx}f(x)\right| + b^2(i2\pi)^{-2}\left|\frac{d}{dx}f(x)\right|^2 \quad (4)$$

Adding and subtracting (2) and (3)

$$\frac{1}{2}\{|L^{-A}[F_{+A}(u)\cdot u](x)|^2 + |L^{+A}[F_{-A}(u)\cdot u](x)|^2\} = a^2|xf(x)|^2 + \frac{b^2}{2\pi}\left|\frac{d}{dx}f(x)\right|^2 \quad (5)$$

And their difference is connected to the amplitude $|f(x)|$and the phase $\varphi(x) = \arg f(x)$

$$\frac{\pi}{ab}\{|L^{-A}[F_{+A}(u)\cdot u](x)|^2 - |L^{+A}[F\_A(u)\cdot u](x)|^2\} = |xf(x)|\cdot\left|\frac{d}{dx}f(x)\right| \quad (6)$$

$$\frac{\pi}{ab}\{|L^{-A}[F_{+A}(u)\cdot u](x)|^2 - |L^{+A}[F\_A(u)\cdot u](x)|^2\} = x|f(x)|^2\frac{d\varphi(x)}{dx} \quad (7)$$

Above eq. (6) can be applied for solving the phase retrieval problem. The phase derivative $\frac{d\varphi}{dx}$, and therefore the phase up to a constant term, can be reconstructed from the knowledge of the intensity $|f(x)|^2$ and the intensity distributions at the output of two Linear Canonical transforms filters with mask u

$$\frac{d\varphi(x)}{dx} = \frac{\pi}{abx|f(x)|^2}\{|L^{-A}[F_{+A}(u)\cdot u](x)|^2 - |L^{+A}[F\_A(u)\cdot u](x)|^2\}$$

Let us consider $R^{-A}[L_A(u)\cdot u^n](x) = \left[du - \frac{bd}{2i\pi}\frac{d}{du}\right]^n$ for the case $n = 2$, which is related to second order derivative of the signal $f(x)$. Filtering in the Linear Canonical domain with mask u^2 gives

$$L^{\mp A}[F_{\pm}(u)\cdot u^2](x) = \left[ax \mp \frac{b}{2\pi i}\frac{d}{dx}\right]^2 f(x)$$

$$= \left[ax \mp \frac{b}{2\pi i}\frac{d}{dx}\right]\left[ax \mp \frac{b}{2\pi i}\frac{d}{dx}\right] f(x)$$

$$= \left[ax \mp \frac{b}{2\pi i}\frac{d}{dx}\right]\left[axf(x) \mp \frac{b}{2\pi i}\frac{d}{dx}f(x)\right]$$

$$= a^2x^2f(x) - \frac{ab}{2\pi i}x\frac{d}{dx}f(x) - \frac{ab}{2\pi i}\left(x\frac{d}{dx}f(x) + f(x)\right) - \frac{b^2}{4\pi^2}\frac{d^2}{dx^2}f(x)$$

$$= \left(a^2x^2 - \frac{ab}{2\pi i}\right)f(x) - \frac{ab}{\pi i}x\frac{d}{dx}f(x) - \frac{b^2}{\pi^2}\frac{d^2}{dx^2}f(x)$$

$$= \left[\left(a^2x^2 - \frac{ab}{2\pi i}\right) - \frac{ab}{\pi i}x\frac{d}{dx} - \frac{b^2}{\pi^2}\frac{d^2}{dx^2}\right]f(x)$$

$$= a^2x^2f(x) - \frac{b^2}{\pi^2}\frac{d^2}{dx^2}f(x) \mp \frac{ab}{2\pi i}\left[2x\frac{d}{dx}f(x) + f(x)\right]$$

Hence adding $L^{-A}[F_A(u)\cdot u^2](x)$ and $L^A[F_{-A}(u)\cdot u^2](x)$ we get

$$\frac{1}{2}\{L^{-A}[F_A(u)\cdot u^2](x) + L^A[F_{-A}(u)\cdot u^2](x)\} = a^2x^2f(x) - \frac{b^2}{\pi^2}\frac{d^2}{dx^2}f(x)$$

3. Conclusion

We hane deduced the general expression for the power filtering of n −th order in Linear Canonical transform domain, which stress its relation to the differentiation operation. The application of linear filtering the Linear Canonical transform domain for phase retrieval from only intensity distribution has been proposed.

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