International Journal of Mathematics Research.
ISSN 0976-5840 Volume 6, Number 3 (2014), pp. 245-251

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Homogeneous Bi-Quadratic Equation With Five Unknowns $2\left(x^4 - y^4\right) = z\left(p^3 + q^3\right)$

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ABSTRACT:

We obtain infinitely many non-zero integer quintuples satisfying the the Biquadratic equation with five unknowns .Various interesting properties among the values of x, y, z,p and q are presented.

KEYWORDS: Biquadratic equation with five unknowns, integral solutions

MSc 2000 Mathematics Subject Classification: 11D25

NOTATIONS:

$$t_{(m,n)} = n\left(1 + \frac{(n-1)(m-2)}{2}\right)$$

$$P_n^m = \left(\frac{n(n+1)}{6}\right)[(m-2)n + (5-m)]$$

$$(OH)_n = \frac{1}{3}n(2n^3+1)$$

$$P_n = n(n+1)$$

INTRODUCTION

The theory of Diophantine equations offers a rich variety of fascinating problems. In particular, biquadratic diophantine equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity [1-5]. In this context, one may refer [6-11] for various problems on the biquadratic diophantine equations with four variables and[12-15] for 5 variables However, often we come across non-homogeneous biquadratic equations and as such one may require its integral solution in its most general form. It is towards this end, this paper concerns with the problem of determining non-trivial integral solutions of the homogeneous equation with five unknowns given by $2\left(x^4 - y^4\right) = z\left(p^3 + q^3\right)$A few relations among the solutions are presented.

METHOD OF ANALYSIS:

The homogeneous bi-quadratic equation with five unknowns to be solved is

$$2\left(x^4 - y^4\right) = z\left(p^3 + q^3\right) \tag{1}$$

Introducing the linear transformations

x= u + v , y = u - v, z= 4v , p= u+ α, q= u- α (2)

in (1) ,it is written as

$$u^2 + 2v^2 = 3\alpha^2 \tag{3}$$

Equation (3) is solved through five different ways and thus, in view of (2), we obtain five different patterns of non-zero distinct integer solutions to (1).

PATTERN :1

Let $\alpha = a^2 + 2b^2$ (4)

Write 3 as

$$3 = \left(1 + i\sqrt{2}\right)\left(1 - i\sqrt{2}\right) \tag{5}$$

Using (4) & (5) in (3) and employing the method of factorization , define

$$u + i\sqrt{2}v = \left(1 + i\sqrt{2}\right)\left(a + i\sqrt{2}b\right)^2$$

Equating real and imaginary parts in the above equation , we get

$$u = a^2 - 2b^2 - 4ab$$
$$v = a^2 - 2b^2 + 2ab$$

Substituting the values of u , v in (2) we have

$$\left.\begin{aligned} x &= 2a^2 - 4b^2 - 2ab \\ y &= -6ab \\ z &= 4a^2 - 8b^2 + 8ab \\ p &= 2a^2 - 4ab \\ q &= -4ab - 4b^2 \end{aligned}\right\} \tag{6}$$

Thus (4)&(6) represent the non-zero distinct integer solutions of (1).

PROPERTIES:

- $x(a,a+1)+2p_a+t_{(6,a)}\equiv 5(\text{mod } 9)$
- $y(a(a+1),a+2)+36p_n^3=0$
- $z(a,4a-3)-8t_{(10,a)}+t_{(250,a)}\equiv -3(\text{mod } 69)$
- $p(a,a+1)-q(a,a+1)-8p_a+2t_{(4,a)}$ is a perfect square.
- $q(a,a+1)+8t_{(3,a)}+t_{(8,a)}\equiv -4(\text{mod } 11)$

PATTERN : 2

Instead of (5) , we write 3as

$$3=\frac{(5+i\sqrt{2})(5-i\sqrt{2})}{3^2} \tag{7}$$

Using (4) & (7) in (3) and employing the method of factorization ,define

$$u+i\sqrt{2}v=\frac{(5+i\sqrt{2})(a+i\sqrt{2}b)^2}{3}$$

Equating the real and imaginary parts in the above equation ,we get

$$u=\frac{1}{3}\left[5a^2-10b^2-4ab\right]$$

$$v=\frac{1}{3}\left[a^2-2b^2+10ab\right]$$

Replacing a by $3A$, b by $3B$ in the above equations& substituting u, v in(2), we get

$$\left.\begin{aligned} x &= 18A^2-36B^2+18AB \\ y &= 12A^2-24B^2-42AB \\ z &= 12A^2-24B^2+120AB \\ p &= 24A^2-12B^2-12AB \\ q &= 6A^2-48B^2-12AB \end{aligned}\right\} \tag{8}$$

Thus (4)&(8) represent the non-zero distinct integer solutions of (1).

PROPERTIES:

- $x(A,A+1)-18p_A+t_{(38,A)}\equiv -36(\text{mod } 89)$
- $y(A,2A^2+1)+126(OH)_A+96t_{(4,A^2)}+t_{(170,A)}\equiv -24(\text{mod } 83)$
- $z(1,B)+t_{(50,B)}\equiv 12(\text{mod } 97)$
- $p(1,B)+t_{(26,B)}\equiv 1(\text{mod } 23)$
- $q(4A-3,A)+12t_{(10,A)}-t_{(98,A)}\equiv 54(\text{mod } 97)$

PATTERN : 3

Write (3) as

$$2v^2 = 3\alpha^2 - u^2 \tag{9}$$

Assume $v = 3a^2 - b^2$ (10)

Write 2 as

$$2 = \left(\sqrt{3}+1\right)\left(\sqrt{3}-1\right) \tag{11}$$

Using (10) & (11) in (9)and employing the method of factorization ,define

$$\sqrt{3}\alpha + u = \left(\sqrt{3}+1\right)\left(\sqrt{3}a + b\right)^2$$

Equating the rational and irrational parts in the above equation, we get

$$\alpha = 3a^2 + b^2 + 2ab$$

$$u = 3a^2 + b^2 + 6ab$$

Substituting the values of u, α in (2) we have

$$\left.\begin{aligned} x &= 6a^2 + 6ab \\ y &= 2b^2 + 6ab \\ z &= 12a^2 - 4b^2 \\ p &= 6a^2 + 2b^2 + 8ab \\ q &= 4ab \end{aligned}\right\} \tag{12}$$

Thus (9)&(12) represent the non-zero distinct integer solutions of (1).

PROPERTIES:

- $x(a^2, a+1) - 6t_{(4,a^2)} - 12p_a^5 = 0$
- $y(a(a+1), a+2) - 36p_a^3 - t_{(6,a)} \equiv 8 \pmod 9$
- $z(a,1) - t_{(26,a)} \equiv -4 \pmod{11}$
- $p(1,b) - t_{(6,b)} \equiv 6 \pmod 9$
- $q(a(a+1), 2a+1) - 24p_a^4 = 0$

PATTERN : 4

(3) is written in the form of ratio as

$$\frac{u+\alpha}{\alpha+v} = \frac{2(\alpha - v)}{u-\alpha} = \frac{a}{b}, b \neq 0 \tag{13}$$

which is equivalent to the system of equations

$$bu - av + (b-a)\alpha = 0 \tag{14}$$

$$au + 2bv - (a + 2b)\alpha = 0 \tag{15}$$

The above system is satisfied by

$$\left.\begin{aligned} u &= a^2 - 2b^2 + 4ab \\ v &= -a^2 + 2b^2 + 2ab \\ \alpha &= a^2 + 2b^2 \end{aligned}\right\} \tag{16}$$

Substituting (16) in (2),the corresponding non- zero distinct integral solutions of (1) are given by

$$\begin{aligned} x &= 6ab \\ y &= 2a^2 - 4b^2 + 2ab \\ z &= -4a^2 + 8b^2 + 8ab \\ p &= 2a^2 + 4ab \\ q &= -4b^2 + 4ab \end{aligned}$$

PROPERTIES:

- $x(a, 2a^2 + 1) - 18(OH)_a = 0$
- $y(a^2, a+1) + t_{(10,a)} - 4p_a^5 - 2t_{(4,a^2)} \equiv -4 \pmod{11}$
- $z(a, a+1) - 8p_a - t_{(10,a)} \equiv 8 \pmod{19}$
- $p(1, b) + q(1, b) + t_{(10,a)} \equiv 2 \pmod 5$
- $q(a(a+1), a+2) - 24p_a^3 + t_{(30,a)} \equiv -16 \pmod{19}$

PATTERN : 5

Also ,(3) is written in the form of ratio as

$$\frac{u + \alpha}{2(\alpha - v)} = \frac{\alpha + v}{u - \alpha} = \frac{a}{b}, b \neq 0 \tag{17}$$

which is equivalent to the system of equations

$$bu + 2av + (b - 2a)\alpha = 0 \tag{18}$$

$$au - bv - (a + b)\alpha = 0 \tag{19}$$

The above system is satisfied by

$$\left.\begin{aligned} u &= -2a^2 + b^2 - 4ab \\ v &= -2a^2 + b^2 + 2ab \\ \alpha &= -2a^2 - b^2 \end{aligned}\right\} \tag{20}$$

Substituting (20) in (2),the corresponding non- zero distinct integral solutions of (1) are given by

$$x = -4a^2 + 2b^2 - 2ab$$
$$y = -6ab$$
$$z = -8a^2 + 4b^2 + 8ab$$
$$p = -4a^2 - 4ab$$
$$q = 2b^2 - 4ab$$

PROPERTIES:

- ❖ $x(a+1, a) + 2p_a - t_{(6,a)} \equiv 5 \pmod 9$
- ❖ $y(a, 2a^2+1) + 18(OH)_a = 0$
- ❖ $z(a^2, a+1) + 8t_{(4,a^2)} - 16p_a^5 - t_{(10,a)} \equiv 4 \pmod{11}$
- ❖ $p(a+1, a) + 8p_a^5 + t_{(10,a)} \equiv -4 \pmod{11}$
- ❖ $q(a, 4a-3) + 4t_{(10,a)} - t_{(66,a)} \equiv 1 \pmod{17}$

Conclusion

In this paper,we have presented different patterns of non-zero distinct integer solutions to the homogeneous biquadratic equation with 5 unknowns $2(x^4 - y^4) = z(p^3 + q^3)$.As the biquadratic diophantine equations are rich in vareity due to the definition of diophantine equations, One may consider biquadratic diophantie equations with unknowns ≥ 5 and search for their integer solutions along with the properties.

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