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Boundedness of Hardy-Steklov operator on weighted Lorentz Spaces $\Lambda_u^p(w)$

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Abstract.

In this paper, we obtain the necessary and sufficient conditions on pair of weights (w_0, w_1) for the boundedness of the Hardy-Steklov operator $\int_{a(x)}^{b(x)} f(t)dt$ between the spaces $\Lambda_u^p(w_0)$ and $\Lambda_v^q(w_1)$ for non-negative functions f, where $1 < p \le q < \infty$.

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1. Introduction

By a weight function u on $(0,\infty)$ we mean a non-negative locally integrable measurable function. We take $\mathrm{M}_0^+ \equiv \mathrm{M}_0^+((0,\infty), u(x)dx)$ to be the set of all those functions which are measurable, non-negative and finite a.e. on $(0,\infty)$ with respect to the measure $u(x)dx$. Then the *distribution function* f_{*_u} of $f \in \mathrm{M}_0^+$ is given by

$$f_{*_u}(\lambda) = \int_{\{x\in(0,\infty):\, f(x)>\lambda\}} u(x)dx, \quad \lambda \ge 0.$$

For a measurable set E, by $u(E)$ we mean $\int_E u(x)dx$.

The *non-increasing rearrangement* f_u^* of f is defined as

$$f_u^*(t) = \inf\{\lambda : f_{*_u}(\lambda) \le t\}, \quad t \ge 0.$$

Essentially, $f_{*_u}^{-1} = f_u^*$ [3].

The *weighted Lorentz spaces* $\Lambda_v^q(w)$ are defined to be the collection of those members of $\mathrm{M}_0^+((0,\infty), v(x)dx)$ for which

$$\|f\|_{\Lambda_v^q(w)} = \begin{cases} \int_0^\infty f_v^{*q}(x)w(x)dx, & 0 < q < \infty \\ \sup_{t>0}\left(\int_0^t w(x)dx\right)^{1/q} f_v^*(t), & q = \infty \end{cases}$$

is finite.

For $w(x) = \frac{q}{p} x^{\frac{q}{p}-1}$, the spaces we get are called *two exponent Lorentz spaces*, denoted as $L_v^{p,q}(0,\infty)$, consisting of $f \in \mathrm{M}_0^+$ for which

$$\|f\|_{L_v^{p,q}(0,\infty)} = \begin{cases} \int_0^\infty \frac{p}{q}[t^{1/p} f_v^*(t)]^q \frac{dt}{t}, & 0 < q < \infty \\ \sup_{t>0} t^{1/p} f_v^*(t), & q = \infty \end{cases}$$

is finite. For $p = q$, the spaces are reduced to *Lebesgue spaces* $L_v^p(0,\infty)$. Again, for $v = 1$, the spaces $\Lambda_v^q(w)$ become $\Lambda^q(w)$, known as *classical Lorentz spaces*. Further, on taking $w(x) = x^{\frac{q}{p}-1}$, we get the so called *two exponent classical Lorentz spaces* $L^{p,q}$. The spaces $\Lambda^q(w)$, may also be looked upon as $L^q(w)$ for the functions f^* *i.e.*, $\|f\|_{\Lambda^q(w)} = \|f^*\|_{L^q(w)}$. And, in case, f is non-increasing, it is precisely $L^q(w)$. In fact, each of the Lorentz spaces talked here, may be deduced from $\Lambda_v^q(w)$, $0 < q \le \infty$. The spaces $\Lambda_v^q(w)$ have been considered for studying the boundedness of various operators. E.g., the boundedness of Hardy Littlewood maximal function on $\Lambda^q(w)$ [6], and on $L_v^{p,q}$ [1]; the boundedness of Hardy operator on $L_v^{p,q}$ [5], and on $\Lambda_v^p(w)$ by Carro and Soria [2] etc.

In the present paper, we study the boundedness of Hardy-Steklov operator on weighted Lorentz spaces. In Section 2, we give some of the known results (from [2], [4] and [6]) which will be used in the subsequent Section 3, containing the main result.

2. Known Results

Definition 1. *The Hardy-Steklov operator is defined as*

$$(Tf)(x) = \int_{a(x)}^{b(x)} f(t)dt$$

where the functions $a = a(x)$ and $b = b(x)$ are strictly increasing and differentiable on $(0,\infty)$. Also, they satisfy

$$a(0) = b(0) = 0; \quad a(\infty) = b(\infty) = \infty \quad \text{and} \quad a(x) < b(x) \text{ for } 0 < x < \infty.$$

Clearly a^{-1} and b^{-1} exist, and are strictly increasing and differentiable as well, for the functions introduced in the definition of Hardy-Steklov operator.

Definition 2. *A function* f *is said to satisfy* Δ_2*-condition if* $f(2x) \le Cf(x)\ \forall\ x > 0$, for some constant $C > 0$.

Theorem A. $\|\cdot\|_{\Lambda_v^q(w)}$ *is a quasi-norm if and only if* $\int_0^x w(t)dt$ *satisfies* Δ_2*-condition.*

Notations.

$$W(y) = \int_{a(x)}^{y} w(t)dt, \quad \tilde{W}(y) = \int_{y}^{b(x)} w(t)dt;$$

$$u^a(y) = u(a^{-1}(y))(a^{-1})'(y), \quad u^b(y) = u(b^{-1}(y))(b^{-1})'(y).$$

We consider now, a sequence $\{m_k\}_{k\in \mathbb{Z}}$ defined as follows:
Let $m > 0$ be fix, set $m_0 = m$, and

$$\left.\begin{array}{rcll} m_{k+1} & = & a^{-1}(b(m_k)), & k \ge 0 \\ m_k & = & b^{-1}(a(m_{k+1})), & k < 0. \end{array}\right\} \tag{1}$$

Obviously $a(m_{k+1}) = b(m_k)\ \ \forall\ \ k \in \mathbb{Z}$.

Lemma 1. *Fix* $m > 0$ *and define* $\{m_k\}_{k\in \mathbb{Z}}$ *as above. Then*

$$m_k < m_{k+1} \text{ for } k \in \mathbb{Z}; \quad \lim_{k\to\infty} m_k = \infty; \quad \text{and} \quad \lim_{k\to-\infty} m_k = 0.$$

Lemma 2. *Let* $p \le q$. *Then a weight* w_0 *satisfies that for every* $\{t_k\}_k \subset \mathbb{R}^+$,

$$\left(\sum_k \left(\int_0^{t_k} w_0(s)ds\right)^{q/p}\right)^{p/q} \le C\int_0^{\sum t_k} w_0(s)ds, \tag{2}$$

if and only if, for every collection of functions $\{f_k\}_k$ in $\Lambda_u^p(w_0)$ with pairwise disjoint support, there exists a constant $C > 0$ such that

$$\sum_k \|f_k\|^q_{\Lambda_u^p(w_0)} \le C \left\|\sum_k f_k\right\|^q_{\Lambda_u^p(w_0)}.$$

A trivial choice of weights for which (2) holds is to take w_0 to be a non-decreasing weight [2].

Theorem B. (Sawyer's Duality Principle) *Let* $1 < p < \infty, v(x)$ *and* $g(x)$ *are non-negative functions on* $[0,\infty[$ *with* v *locally integrable. Then*

$$\sup_{\downarrow f \ge 0} \frac{\int_0^\infty f(x)g(x)dx}{(\int_0^\infty f^p(x)v(x)dx)^{1/p}}$$
$$\approx \left(\int_0^\infty \left(\int_0^x g\right)^{p'} \left(\int_0^x v\right)^{-p'} v(x)dx\right)^{1/p'} + \left(\int_0^\infty g\right)\left(\int_0^\infty v\right)^{-1/p}$$

where the symbol $\approx$ means that the ratio of left and right hand sides is bounded between two positive constants depending only on p (and not on v or g).

3. Main Result

Theorem. *Let* $1 < p \le q < \infty$, $x < a(x)$ *for* $0 < x < \infty$, $W_1 \in \Delta_2$ *and* w_0 *be satisfying the condition (2). Then* $T : \Lambda_u^p(w_0) \to \Lambda_v^q(w_1)$, *i.e., the inequality*

$$\left(\int_0^\infty \left(\int_{a(x)}^{b(x)} f(t)dt\right)_v^{*q} w_1(x)dx\right)^{1/q} \le C\left(\int_0^\infty f_u^{*p}(x)w_0(x)dx\right)^{1/p} \quad (3)$$

holds if and only if $\forall \ f \ge 0$

$$\sup_{a(x)<y<b(x)} [\int_{a(x)}^{b(x)} \left(\int_{a(x)}^{y} (\chi_{(a(x),y)}u^{-1})_u^*(t)dt\right)^{p'} W_0(y)^{-p'} w_0(y)dy$$
$$+ (y - a(x))W_0(b(x))^{1/p}][W_1(\Phi(y))]^{1/q} < \infty; \quad (4)$$

where

$$\Phi(y)=\begin{cases} U_1(y) & \text{when } W_1(z)=\int_{a(x)}^{z} w_1^b(s)ds \\ \tilde{U}_1(y) & \text{when } W_1(z)=\int_{a(x)}^{z} w_1^a(s)ds. \end{cases}$$

Proof. $(\Leftarrow)$ (Sufficiency part)

Fix $m>0$ and define $\{m_k\}_{k\in Z}$ as in (1). Denote $E_k=(m_k,m_{k+1})$, and $a_k=a(m_k), b_k=b(m_k)$. Then for $x\in E_k$, $a(x)<a_{k+1}=b_k<b(x)$. Now since W_1 satisfies Δ_2-condition, we have

$$\left(\int_0^\infty \left(\int_{a(x)}^{b(x)} f(t)dt\right)_v^{*q} w_1(x)dx\right)^{1/q}$$

$$=\left(\int_0^\infty \left(\sum_{k\in Z}\chi_{E_k}(x)\int_{a(x)}^{b_k} f(t)dt+\sum_{k\in Z}\chi_{E_k}(x)\int_{a_{k+1}}^{b(x)} f(t)dt\right)_v^{*q} w_1(x)dx\right)^{1/q}$$

$$\le C\left[\underbrace{\left(\int_0^\infty \left(\sum_{k\in Z}\chi_{E_k}(x)\int_{a(x)}^{b_k} f(t)dt\right)_u^{*q} w_1(x)dx\right)^{1/q}}_{I_2} + \underbrace{\left(\int_0^\infty \left(\sum_{k\in Z}\chi_{E_k}(x)\int_{a_{k+1}}^{b(x)} f(t)dt\right)_v^{*q} w_1(x)dx\right)^{1/q}}_{I_1}\right]. \qquad (5)$$

Now notice that, for $x=m_{k+1}$ the condition (4) becomes

$$\sup_{a_{k+1}<y<b_{k+1}}\left[\int_{a_{k+1}}^{b_{k+1}}\left(\int_{a_{k+1}}^{y}(\chi_{(a_{k+1},y)}u^{-1})_u^*(t)dt\right)^{p'}\left(\int_{a_{k+1}}^{y} w_0(t)dt\right)^{-p'} w_0(y)dy + (y-a_{k+1})\left(\int_{a_{k+1}}^{b_{k+1}} w_0(t)dt\right)^{1/p}\right]\left(\int_{a_{k+1}}^{\int_{a_{k+1}}^{y}u_1(t)dt} w_1^b(s)ds\right)^{1/q}<\infty \quad (6)$$

which is a sufficient condition for the inequality

$$\left(\int_{a_{k+1}}^{b_{k+1}}\left(\int_{a_{k+1}}^{y} f(t)dt\right)_v^{*q} w_1^b(y)dy\right)^{1/q}\le C\left(\int_{a_{k+1}}^{b_{k+1}} f_u^{*p}(x)w_0(x)dx\right)^{1/p} \qquad (7)$$

to hold. Thus, in view of (6), (7) and Lemma 2, we have

$$
\begin{aligned}
I_1 &= \left(\sum_{k\in Z} \int_{m_k}^{m_{k+1}} \left(\int_{a_{k+1}}^{b(x)} f(t)dt \right)_v^{*q} w_1(x)dx \right)^{1/q} \\
&= \left(\sum_{k\in Z} \int_{a_{k+1}}^{b_{k+1}} \left(\int_{a_{k+1}}^{y} f(t)dt \right)_v^{*q} w_1^b(y)dy \right)^{1/q} \\
&\le C \left(\sum_{k\in Z} \left(\int_{a_{k+1}}^{b_{k+1}} f_u^{*p}(x) w_0(x)dx \right)^{q/p} \right)^{1/q} \\
&= C \left(\sum_{k\in Z} \left(\int_0^{\infty} (f_k)_u^{*p}(x) w_0(x)dx \right)^{q/p} \right)^{1/q} \quad \text{for } f_k = \chi_{(a_{k+1},b_{k+1})} f \\
&\le C \left(\int_0^{\infty} (\sum_{k\in Z} f_k)_u^{*p}(x) w_0(x)dx \right)^{1/p} \\
&= C \left(\int_0^{\infty} f_u^{*p}(x) w_0(x)dx \right)^{1/p}. \qquad (8)
\end{aligned}
$$

Again, on taking $x = m_k$ in (4), we obtain the expressions conjugate to (6) and (7). And then, further, using Lemma 2, we get

$$
\begin{aligned}
I_2 &= \left(\sum_{k\in Z} \int_{m_k}^{m_{k+1}} \left(\int_{a(x)}^{b_k} f(t)dt \right)_v^{*q} w_1(x)dx \right)^{1/q} \\
&= \left(\sum_{k\in Z} \int_{a_k}^{b_k} \left(\int_{y}^{b_k} f(t)dt \right)_v^{*q} w_1^b(y)dy \right)^{1/q} \\
&\le C \left(\sum_{k\in Z} \left(\int_{a_k}^{b_k} f_u^{*p}(x) w_0(x)dx \right)^{q/p} \right)^{1/q} \\
&\le C \left(\int_0^{\infty} f_u^{*p}(x) w_0(x)dx \right)^{1/p}. \qquad (9)
\end{aligned}
$$

The inequality (3), now follows from (8) and (9).

($\Rightarrow$) (Necessary part)

It can be easily verified that if the condition (2) is true for a weight w_0, then so is this for $\tilde{w}_0 = w_0 \chi_{(a(x),b(x))}$, $x \in (0,\infty)$. So, let $f \in \Lambda_u^p(\tilde{w}_0)$, Then, for every $t \in [y, b(x)]$, we have

$$\int_{a(x)}^{y} f(s)ds \le T(\chi_{(a(x),y)} f)(t).$$

Let $\eta < \int_{a(x)}^{y} f(s)ds$, then

$$\int_{y}^{b(x)} v(s)ds \le [T(\chi_{(a(x),y)} f)]_{*_v}(\eta).$$

Thus,

$$\begin{aligned}
\eta\left(\int_{a(x)}^{\int_y^{b(x)} v(s)ds} w_1^a(s)ds\right)^{1/q} &= \eta\left(\int_{x}^{a^{-1}(\int_y^{b(x)} v(s)ds)} w_1(z)dz\right)^{1/q} \\
&\le \eta\left(\int_{x}^{\int_y^{b(x)} v(s)ds} w_1(z)dz\right)^{1/q} \\
&\le \eta\left(\int_{x}^{[T(\chi_{(a(x),y)} f)]_{*_v}(\eta)} w_1(z)dz\right)^{1/q} \\
&\le \sup_{z>0} z\left(\int_{[T(\chi_{(a(x),y)} f)]_{*_v}(z)} w_1(s)ds\right)^{1/q} \\
&= \left\|T(f\chi_{(a(x),y)})\right\|_{\Lambda_v^{q;\infty}(w_1)} \\
&\le \left\|T(f\chi_{(a(x),y)})\right\|_{\Lambda_v^{q}(w_1)} \\
&\le C\|f\|_{\Lambda_u^p(\tilde{w}_0)}.
\end{aligned}$$

Now, on taking the supremum over all $\eta < \int_{a(x)}^{y} f(s)ds$, and using the Sawyer's duality principle, we obtain the condition (4) with $W_1(z) = \int_{a(x)}^{z} w_1^a(s)ds$.

Again for every $t \in [a(x), y]$, we have

$$\int_{y}^{b(x)} f(s)ds \le T(\chi_{(y,b(x))} f)(t),$$

and therefore, on taking $\eta < \int_{y}^{b(x)} f(s)ds$, we get

$$\int_{a(x)}^{y} v(s)ds \le \int_{\{s:T(\chi_{(y,b(x))} f)(s)>\eta\}} v(s)ds = [T(\chi_{(y,b(x))} f)]_{*_v}(\eta).$$

Thus

$$\eta\left(\int_{a(x)}^{\int_{a(x)}^{y} v(s)ds} w_1^b(s)ds\right)^{1/q} \le \eta\left(\int_{x}^{\int_{a(x)}^{y} v(s)ds} w_1(z)dz\right)^{1/q}$$

$$\leq \sup_{z>0} z\left(\int_0^{[T(\chi_{(y,b(x))}f)]^*_v(z)} w_1(s)ds\right)^{1/q} \leq C\|f\|_{\Lambda_u^p}(\tilde{w}_0).$$

Taking supremum over all those $\eta < \int_y^{b(x)} f(s)ds$, we have

$$\sup_{a(x)<y<b(x)}\left(\sup_f \frac{\int_{a(x)}^y f(t)dt}{\|f\|_{\Lambda_u^p(\tilde{w}_0)}}\right)\left(\int_{a(x)}^{\int_{a(x)}^y u_1(s)ds} w_1^b(s)ds\right)^{1/q} < \infty .$$

And, now using Sawyer's duality principle, the condition (4) holds with $W_1(z) = \int_{a(x)}^z w_1^b(s)ds$.

Remark. *It is known that, in particular, for the two exponent classical Lorentz spaces the following embeddings hold when* $p \leq q$ *:*

$$L^{p,1} \subset \ldots \subset L^p \subset \ldots \subset L^{p,q} \subset \ldots \subset L^{p,\infty}.$$

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References

[1] H. M. Chung, R. A. Hunt and D. S. Kurtz, *The Hardy-Littlewood maximal function on* $L(p,q)$ *spaces with weights*, Indiana University Mathematics Journal, 31(1), 109-120, 1982.

[2] M. J. Carro and J. Soria, *Weighted Lorentz spaces and Hardy operator*, J. Func. Anal., 112(2), 480–494, 1993.

[3] R. A. Hunt, *On* $L(p,q)$ *spaces*, Enseign Math (2), 12, 249-276, 1966.

[4] A. Kufner and L.E. Persson, *Weighted Inequalites of Hardy Type*, World Scientific Publishing Co. Pte. Ltd, Singapore, 2003.

[5] E. Sawyer, *Weighted Lebesgue and Lorentz norm inequalites for the Hardy operator*, Trans. of Amer. Math. Soc., Vol. 281(1), 329-337, 1984.

[6] E. Sawyer, *Boundedness of classical operators on classical Lorentz spaces*, Studia Mathematica, T, XCVI, 145-158, 1990.