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Inequalities Among Related Pairs Of Lucas Numbers

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ABSTRACT

In this paper we consider Lucas inequalities and relate them through the sequence $\{u_r\}_{r=0}^{n}$ defined by $u_r = L_r L_{n+1-r}$ where n is a fixed natural number and $L_1, L_2, L_3, ...$ are the ordinary Lucas numbers.

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1. INTRODUCTION

"Fibonacci inequalities" have been studied in a variety of contexts. Atanassov [3] considered the Fibonacci inequalities and relate them through the sequence $\{m_r\}_{r=0}^{n}$ defined by $m_r = F_r F_{n+1-r}$ where n is a fixed natural number and $F_1, F_2, F_3, ...$ are ordinary Fibonacci numbers as defined in [4]. Here we consider the Lucas inequalities and relate them through the sequence $\{u_r\}_{r=0}^{n}$ defined by $u_r = L_r L_{n+1-r}$ where n is a fixed natural number and $L_1, L_2, L_3, ...$ are the ordinary Lucas numbers [4].

2. MAIN RESULT

Theorem: For every natural number k, the following inequalities for the elements of the sequence $\{u_r\}_{r=1}^{n}$ are valid:

(a) if n = 4k, then

$$L_1L_{4k} < L_3L_{4k-2} < \ldots < L_{2k-1}L_{2k+2} < L_{2k}L_{2k+1} < L_{2k-2}L_{2k+3} < \ldots < L_2L_{4k-1}$$

(b) if n = 4k+1, then

$$L_1L_{4k+1} < L_3L_{4k-1} < \ldots < L_{2k-1}L_{2k+3} < L_{2k+1}L_{2k+1} < L_{2k}L_{2k+2} < \ldots < L_2L_{4k}$$

(c) if n = 4k+2, then

$$L_1L_{4k+2} < L_3L_{4k} < \ldots < L_{2k+1}L_{2k+2} < L_{2k}L_{2k+3} < L_{2k-2}L_{2k+5} < \ldots < L_2L_{4k+1}$$

(d) if n = 4k+3, then

$$L_1L_{4k+3} < L_3L_{4k+1} < \ldots < L_{2k+1}L_{2k+3} < L_{2k+2}L_{2k+2} < L_{2k}L_{2k+4} < \ldots < L_2L_{4k+2}$$

Examples when k = 3:

(a) n = 12,

$L_1L_{12} = 322, L_3L_{10} = 492, L_5L_8 = 517, L_6L_7 = 522, L_4L_9 = 532, L_2L_{11} = 597$.

which gives $L_1L_{12} < L_3L_{10} < L_5L_8 < L_6L_7 < L_4L_9 < L_2L_{11}$

(b) n = 13,

$L_1L_{13} = 521, L_3L_{11} = 796, L_5L_9 = 836, L_7L_7 = 841, L_6L_8 = 846, L_4L_{10} = 861,$
$L_2L_{12} = 966$

which gives $L_1L_{13} < L_3L_{11} < L_5L_9 < L_7L_7 < L_6L_8 < L_4L_{10} < L_2L_{12}$

(c) n = 14,

$L_1L_{14} = 843, L_3L_{12} = 1288, L_5L_{10} = 1353, L_7L_8 = 1363, L_6L_9 = 1368,$
$L_4L_{11} = 1393, L_2L_{13} = 1563$

which gives $L_1L_{14} < L_3L_{12} < L_5L_{10} < L_7L_8 < L_6L_9 < L_4L_{11} < L_2L_{13}$

(d) n = 15,

$L_1L_{15} = 1364, L_3L_{13} = 2084, L_5L_{11} = 2189, L_7L_9 = 2204, L_8L_8 = 2209,$
$L_6L_{10} = 2214, L_4L_{12} = 2254, L_2L_{14} = 2529,$

which gives $L_1L_{15} < L_3L_{13} < L_5L_{11} < L_7L_9 < L_8L_8 < L_6L_{10} < L_4L_{12} < L_2L_{14}$.

Proof: We shall use induction method to prove the required results.
We shall prove the case (a) and other cases can be proved in a similar manner.
For k = 1, $L_1L_4 = 7 < L_3L_2 = 12$
the result is true.
So, by induction method assume that the case (a) is true for some $k \geq 1$ and then prove it for k+1.
First, we see that

$$L_1L_{4k+4} - L_3L_{4k+2} = L_{4k+1} - 2L_{4k+2} < 0$$

This shows that $L_1L_{4k+4} < L_3L_{4k+2}$

i.e., the inequality $L_{2i-1}L_{4k-2i+6} < L_{2i+1}L_{4k-2i+4}$ ...(2.1)

is valid for i = 1.
Let us assume that for some $i, 1 \leq i \leq k$, the inequality (2.1) is true. Then we must prove that

the inequality $L_{2i+1}L_{4k-2i+4} < L_{2i+3}L_{4k-2i+2}$ ...(2.2)
is also true.
But,

$$\begin{aligned} & L_{2i+1}L_{4k-2i+4} - L_{2i+3}L_{4k-2i+2} \\ &= L_{2i+1}\left(L_{4k-2i+3} + L_{4k-2i+2}\right) - \left(L_{2i+2} + L_{2i+1}\right)L_{4k-2i+2} \\ &= L_{2i+1}L_{4k-2i+3} - L_{2i+2}L_{4k-2i+2} < 0 \end{aligned}$$

By the inductive assumption of (d). Therefore, inequality (2.2) is true and hence (a) is true.

3. SUPPLEMENT RESULT

With the help of extremal problems discussed by Atanassov in [2], we can write the following:

Corollary: For every natural number n the maximal and minimal element of the sequence $\{u_r\}_{r=0}^{n}$ is L_0L_{n+1} *and* L_1L_n respectively.

4.CONCLUSION

This paper discuss the Lucas inequalities through the sequence $\{u_r\}_{r=0}^{n}$ and its maximal and minimal element. The new results can be obtained by defining a sequence in different manner.

5. REFERENCES

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