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Fibonacci and Lucas Polynomial Identities, Binomial coefficients and Pascal's Triangle

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ABSTRACT

In this paper we will prove some generalized identities involving Fibonacci and Lucas polynomials and for rapid numerical calculation of identities we present each identity as summation involving binomial coefficients.

Keywords: Fibonacci Polynomial, Lucas Polynomial, Binomial coefficient, Pascal's Triangle and Pascal's identity.

INTRODUCTION

The most prominent linear recurrence relation of order two with variable coefficient is one that defines Fibonacci polynomials [5], it is define recursively as:

$$F_{n+1}(x) = xF_n(x) + F_{n-1}(x) \quad ,n \geq 1 \text{ with } F_0(x) = 0 \text{ and } F_1(x) = 1 \qquad (1)$$

Binet's Formula for Fibonacci polynomial

The well known Binet's formula allows us to express the n^{th}Fibonacci Polynomial in function of the roots $r_1 = \alpha$ and $r_2 = \beta$ of the characteristic equation $r^2 - xr - 1 = 0$ associated to the recurrence relation (1) as:

$$F_n(x) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \tag{2}$$

Lucas Polynomial [5]

Another prominent linear recurrence relation of order two with variable coefficient is defined by Lucas Polynomial as:

$$L_{n+1}(x) = xL_n(x) + L_{n-1}(x) \quad ,n \geq 1 \text{ with } L_0(x) = 2 \text{ and } L_1(x) = x \tag{3}$$

Binet's Formula for Lucas polynomial

The Binet's formula allows us to express the n^{th}Lucas polynomial in function of the roots $r_1 = \alpha$ and $r_2 = \beta$ of the characteristic equation $r^2 - xr - 1 = 0$ associated to the recurrence relation (3) as:

$$L_n(x) = \alpha^n + \beta^n \tag{4}$$

Finding the exact expression of $F_n(x)$ from equation (2) requires multiple steps of busy and messy algebraic calculation which is not desirable, so in this paper we present $F_n(x)$ as a summation involving binomial coefficient for quick numerical calculation. Likewise we use this summation to write some fundamental identities concerning Fibonacci and Lucas polynomials and develop some new identities using Fibonacci and Lucas Polynomials.

Fibonacci Polynomial, Pascal's triangle and Binomial Coefficient [2]

The well known Pascal's triangle shown in Table 1 is one of the world's most recognized number patterns.

Table 1

1				
1	1			
1	2	1		
1	3	3	1	
1	4	6	4	1

Its entries can be presented as binomial coefficient, describe the expansion of $(x+y)^n$ for any non-negative integer n. In particular the k^{th} entry along n^{th} row of Pascal's triangle corresponds to the coefficient of $x^k\, y^{n-k}$ and is given by the well known factorial formula:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, (0 \le k \le n) \tag{5}$$

The most celebrated property of Pascal's triangle is its triangular recurrence given by:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \tag{6}$$

which express each binomial coefficient as a summation of the two entries immediately above it, moreover this recurrence uniquely define Pascal's triangle, if we initialize the boundary value along its outer diagonals to 1 that is

$$\binom{n}{0} = \binom{n}{n} = 1 \tag{7}$$

all other entries are then generated using (6).

Table 2: Pascal's triangle as Binomial coefficient

$\binom{0}{0}$

$\binom{1}{0}$ $\binom{1}{1}$

$\binom{2}{0}$ $\binom{2}{1}$ $\binom{2}{2}$

$\binom{3}{0}$ $\binom{3}{1}$ $\binom{3}{2}$ $\binom{3}{3}$

…

Table 3: First few Fibonacci Polynomials by (1)

$F_1(x) = 1$

$F_2(x) = x$

$F_3(x) = x^2 + 1$

$F_4(x) = x^3 + 2x$

$F_5(x) = x^4 + 3x^2 + 1$

$F_6(x) = x^5 + 4x^3 + 3x$

…

We may present coefficient of each term of Fibonacci Polynomials as binomial coefficients first few polynomials are presented in Table 4:

Table 4

$F_1(x) = \binom{0}{0}$

$F_2(x) = \binom{1}{0}x$

$F_3(x) = \binom{2}{0}x^2 + \binom{1}{1}$

$F_4(x) = \binom{3}{0}x^3 + \binom{2}{1}x$

$F_5(x) = \binom{4}{0}x^4 + \binom{3}{1}x^2 + \binom{2}{2}$

$F_6(x) = \binom{5}{0}x^5 + \binom{4}{1}x^3 + \binom{3}{2}x$

…

$F_{n+1}(x) = \binom{n}{0}x^n + \binom{n-1}{1}x^{n-2} + \cdots + \binom{n-\lfloor\frac{n}{2}\rfloor}{\lfloor\frac{n}{2}\rfloor}x^{n-2\lfloor\frac{n}{2}\rfloor}$

So by inspection we can see that for any integer $n \geq 0$

$$F_{n+1}(x) = \sum_{i=0}^{\lfloor\frac{n}{2}\rfloor} \binom{n-i}{i} x^{n-2i} \tag{8}$$

where $\lfloor n \rfloor$ represent the floor function.

Now using equations (1), (6), (7) and Table 4

$$F_{n+2}(x) = xF_{n+1}(x) + F_n(x)$$

$$= x\left[\binom{n}{0}x^n + \binom{n-1}{1}x^{n-2} + \cdots + \binom{n-\lfloor\frac{n}{2}\rfloor}{\lfloor\frac{n}{2}\rfloor}x^{n-2\lfloor\frac{n}{2}\rfloor}\right] + \left[\binom{n-1}{0}x^{n-1} + \binom{n-2}{1}x^{n-3} + \cdots + \binom{n-1-\lfloor\frac{n-1}{2}\rfloor}{\lfloor\frac{n-1}{2}\rfloor}x^{n-1-2\lfloor\frac{n-1}{2}\rfloor}\right]$$

$$= \left[\binom{n+1}{0}x^{n+1} + \binom{n}{1}x^{n-1} + \cdots + \binom{n+1-\lfloor\frac{n+1}{2}\rfloor}{\lfloor\frac{n+1}{2}\rfloor}x^{n+1-2\lfloor\frac{n+1}{2}\rfloor}\right]$$

i.e.

$$F_{n+2}(x) = \left[\binom{n+1}{0}x^{n+1} + \binom{n}{1}x^{n-1} + \cdots + \binom{n+1-\lfloor\frac{n+1}{2}\rfloor}{\lfloor\frac{n+1}{2}\rfloor}x^{n+1-2\lfloor\frac{n+1}{2}\rfloor}\right]$$

where $\lfloor n \rfloor$ represent the floor function.

From above discussion it is clear that each Fibonacci polynomial can be represented using equation (8).

As result of above discussion and the definition of Fibonacci and Lucas polynomials we obtained the following theorems

Theorem 1.1[3]: If $F_n(x)$ is Fibonacci Polynomial then for any integer $n \geq 0$:

(i)

$$F_n^2(x) + F_{n+1}^2(x) = \sum_{i=0}^{n} x^{2n-2i} \binom{2n-i}{i}$$

(ii)

$$F_{n+2}^2(x) - F_n^2(x) = \sum_{i=0}^{\lfloor \frac{2n+1}{2} \rfloor} x^{2n+2-2i} \binom{2n+1-i}{i}$$

Theorem 1.2[3]: If $F_n(x)$ is Fibonacci Polynomial then for any integer $n \geq 0$:

(i)

$$F_{n+2}(x)F_{n+3}(x) - F_n(x)F_{n+1}(x) = \sum_{i=0}^{n+1} x^{2n+3-2i} \binom{2n+2-i}{i}$$

(ii)

$$xF_{n+1}^3(x) + F_{n+2}^3(x) - F_n^3(x) = \sum_{i=0}^{\lfloor \frac{3n+2}{2} \rfloor} x^{3n+3-2i} \binom{3n+2-i}{i}$$

Theorem 1.3[4]: If $F_n(x)$ and $L_n(x)$ are Fibonacci and Lucas Polynomials respectively then for any integer $n \geq 0$:

(i)

$$\frac{1}{(x^2+4)} [L_{n+1}(x) + L_{n+3}(x)] = \sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} x^{n+1-2i} \binom{n+1-i}{i}$$

(ii)

$$\frac{1}{(x^2+4)} [L_{n+5}(x) - L_{n+1}(x)] = \sum_{i=0}^{\lfloor \frac{n+2}{2} \rfloor} x^{n+3-2i} \binom{n+2-i}{i}$$

(iii)

$$\frac{1}{(x^2+4)}\left[L_{n+1}^2(x)+L_{n+2}^2(x)\right]=\sum_{i=0}^{n+1} x^{2n+2-2i}\binom{2n+2-i}{i}$$

Theorem 1.4[4]: If $F_n(x)$ and $L_n(x)$ are Fibonacci and Lucas Polynomials respectively then for any integer $n \geq 0$:

(i)

$$\frac{1}{(x^2+4)}\left[2L_{2n+2}(x)-L_{n+1}^2(x)\right]=\left[\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor} x^{n-2i}\binom{n-i}{i}\right]^2$$

(ii)

$$\frac{1}{(x^2+4)}\left[L_{n+1}(x)L_{n+3}(x)+(-1)^n x^2\right]=\left[\sum_{i=0}^{\lfloor\frac{n+1}{2}\rfloor} x^{n+1-2i}\binom{n+1-i}{i}\right]^2$$

Theorem 1.5: If $F_n(x)$ and $L_n(x)$ are Fibonacci and Lucas Polynomials respectively then for any integer $n \geq 0$:

(i)

$$L_{2n+2}(x)\left[\sum_{i=0}^{n+1} L_{2n+4i}(x)\right]-L_{4n+2}(x)=x^2+2+\sum_{i=0}^{\lfloor\frac{8n+7}{2}\rfloor} x^{8n+6-2i}\binom{8n+7-i}{i}$$

(ii)

$$\prod_{i=0}^{n} L_{2^i}(x)=\sum_{i=0}^{\lfloor\frac{2^{n+1}-1}{2}\rfloor} x^{2^{n+1}-1-2i}\binom{2^{n+1}-1-i}{i}$$

Proof: To prove Theorems 1.1-1.5 we can simply using equations (1), (3), (8) and the fact that each expression on the left hand side can be written as a single or power of Fibonacci Polynomial, we could use the principal of mathematical induction, Binet's formula, combinatorial arguments or just simple algebra to prove the theorems.

References

[1]. Verner E. Hoggatt and Gerald E. Bergum, A Problem of Fermat and the Fibonacci Sequence, The Fibonacci Quarterly, Vol. 15, No. 4, Oct. 1977, pp. 323-330.

[2]. M.K. Azarian, Fibonacci Identities as Binomial Sums, International Journal of Contemporary Mathematical Sciences, Vol. 7, 2012, No. 38, 1871-1876.

[3]. M.K. Azarian, Fibonacci Identities as Binomial Sums II, International Journal of Contemporary Mathematical Sciences, Vol. 7, 2012,No. 42, 2053-2059.

[4]. M.K. Azarian, Identities Involving Lucas or Fibonacci and Lucas Numbers as Binomial Sums, International Journal of Contemporary Mathematical Sciences, Vol. 7,2012, No. 45, 2221-2227.

[5]. Zhengang Wu and Wenpeng Zang, Several Identities Involving the Fibonacci Polynomials and Lucas Polynomials, Journal of Inequalities and Applications, 2013, 2013:205.