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Selection Rejection Methodology For Three Dimensional Continuous Random Variables And Its Application To Three Dimensional Normal Distribution

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ABSTRACT

In this paper we have generalized the Selection-Rejection Methodology for two dimensional continuous random variables to three dimensional continuous random variables and applied it to the three dimensional normal distribution.

Keywords: - random variable, iterations, target probability distribution and proposal probability distribution.

1. INTRODUCTION.

The Selection-Rejection Methodology for one dimensional continuous random variables was developed based on the idea of Acceptance-Rejection Method by the renowned mathematician, Von Neumann, from the University of Berlin. Von Neumann[1] came forward with his method during 1950's but later on Karl Sigman[2] from Columbia University gave the similar methodology in 2007.Again in 1989,Bernard D.Flury[3] from Indiana University came forward with the theory "Acceptance-Rejection Sampling Made Easy".D.P.Kroese[4] from University of Queensland put forward his theory of Acceptance-Rejection in 2011.Selection-Rejection Methodology can be applied to almost all statistical distributions and hence it has got immense physical significance.

2. SELECTION-REJECTION METHODOLOGY FOR THREE DIMENSIONAL CONTINUOUS RANDOM VARIABLES.

Let X,Y,Z be a two dimensional continuous random variable with probability distribution function $f \;\; x,y,z \;\; \forall x,y,z \in R$,where R=set of all real numbers. Let $g \;\; x,y,z \;\; \forall x,y,z \in R$ where R=set of all real numbers be another probability density function such that $\frac{f(x,y,z)}{g(x,y,z)} \le k \;\; \forall\, (x,y,z) \in R$, where $k \ge 1$ is a real number. By successively selecting different values of (X,Y,Z) we will try to make the ratio $\frac{f(x,y,z)}{kg(x,y,z)}$ as close to 1 as possible. The probability density function $f(x,y,z)$ is called target distribution and he probability density function $g(x,y,z)$ is called proposal distribution.

The step by step procedure for the Selection-Rejection Methodology is as follows.

Step (1):- Let (X,Y,Z) be a three dimensional continuous random variable with probability distribution function $f(x,y,z) \;\; \forall x,y,z \in R$, where R=set of all real numbers.

Step (2):- Let (X',Y',Z') be another three dimensional continuous random variable (which is independent of (X,Y,Z)) with probability distribution function $g(x,y,z)$ $\forall x,y,z \in R$, where R=set of all real numbers.

Step (3):- Let $\frac{f(X',Y',Z')}{g(X',Y',Z')} \le k \;\; \forall\, (X',Y',Z') \in R$, where $k \ge 1$ a real number.

Step (4):- Let $0 < R_1 < 1$, $0 < R_2 < 1$ and $0 < R_3 < 1$ be three random numbers.

Step (5):- Set X' in terms of R_1, set Y' in terms of R_2 and set Z' in terms of R_3 depending on the expression obtained for the ratio $\frac{f(X',Y',Z')}{kg(X',Y',Z')}$.

Step (6):- If $R_1R_2R_3 \le \frac{f(X',Y',Z')}{kg(X',Y',Z')}$, then set $(X,Y,Z) = (X',Y',Z')$ and select the continuous random variable (X',Y',Z'); otherwise reject the variable (X',Y',Z') and repeat the process from step (1).

The probability that the continuous random variable (X',Y',Z') is selected is $\frac{1}{k}$.

The number of iterations required to select (X',Y',Z') is k.

It may be noted that $0 \le \frac{f(X',Y',Z')}{kg(X',Y',Z')} \le 1$

To prove that the probability for the selection of (X',Y',Z') **is** $\frac{1}{k}$**,**

Proof: - $P\left(Select \,\middle|\, (X',Y',Z')\right) = P\left(R_1R_2R_3 \le \frac{f(X',Y',Z')}{kg(X',Y',Z')}\right) = \frac{f(X',Y',Z')}{kg(X',Y',Z')}$..

$$P\left(X',Y',Z' \text{ is selected}\right)=\int_{-\infty}^{x}\int_{-\infty}^{y}\int_{-\infty}^{z}\frac{f(w,\mathrm{u},\mathrm{v})}{kg(w,\mathrm{u},\mathrm{v})}g(w,\mathrm{u},\mathrm{v})\,dwdudv$$

$$=\frac{1}{k}\int_{-\infty}^{x}\int_{-\infty}^{y}\int_{-\infty}^{z}f(w,\mathrm{u},v)\,dwdudv=\frac{1}{k}\quad\left[\because\int_{-\infty}^{x}\int_{-\infty}^{y}\int_{-\infty}^{z}f(w,\mathrm{u},v)\,dwdudv=1\right]$$

Hence the proof.

Since the probability of selection (i.e. success) is $\frac{1}{k}$,the number of iterations needed will follow a geometric distribution with $p=\frac{1}{k}$.So, on average it will take k iterations to generate a number.

3. APPLICATION TO THREE DIMENSIONAL NORMAL DISTRIBUTION.

Two dimensional normal distribution is given by

$$f(x,y,z)=\frac{1}{\sqrt{2\pi}}e^{-\left(\frac{x^2}{2}+\frac{x^2}{2}+\frac{z^2}{2}\right)}\quad ,x\geq 0,y\geq 0,\mathrm{z}\geq 0,x,y,z\in R \tag{1}$$

Here $f(x,y,z)$ is the target distribution.

Let $g(x,y,z)=e^{-(x+y+z)}$ $,x\geq 0,y\geq 0,z\geq 0$ be the proposal distribution. (2)

Let $h(x,y,z)=\dfrac{f(x,y,z)}{g(x,y,z)}=\dfrac{1}{\sqrt{2\pi}}\exp\left(-\left(\dfrac{x^2+y^2+z^2-2x-2y-2z}{2}\right)\right)$ (3)

With the help of differential calculus we can show that $h(x,y,z)$ attains maximum at $(1,1,1)$ and the maximum value of $h(x,y,z)$ is

$$\frac{e^{\frac{3}{2}}}{\sqrt{2\pi}}\approx 1.7879\,(approximately).$$

Choosing $k=\dfrac{e^{\frac{3}{2}}}{\sqrt{2\pi}}$, we get

$$\frac{f(x,y,z)}{kg(x,y,z)}=\exp\left(-\frac{(x-1)^2}{2}\right)\times\exp\left(-\frac{(y-1)^2}{2}\right)\times\exp\left(-\frac{(z-1)^2}{2}\right) \tag{4}$$

Selection-Rejection Methodology for the two dimensional distribution is as follows

Step (1):- Let X, Y, Z be a two dimensional continuous random variable with probability distribution function $f\ x, y, z\ \ \forall x, y, z \in R$, where R=set of all real numbers.

Step (2):- Let X', Y', Z' be a two dimensional continuous random variable with probability distribution function $g\ x, y, z\ \ \forall x, y, z \in R$, where R=set of all real numbers.

Step (3):-Let $0 < R_1 < 1$, $0 < R_2 < 1$ and $0 < R_3 < 1$ be three random numbers.

Step (4):- Set $X' = 1 + \sqrt{-2\ln(R_1)}$,$Y' = 1 + \sqrt{-2\ln(R_2)}$, and $Z' = 1 + \sqrt{-2\ln(R_3)}$

Step (5):-If $R_1 R_2 R_3 \le \exp\left(-\dfrac{X'-1^{\ 2}}{2}\right) \times \exp\left(-\dfrac{Y'-1^{\ 2}}{2}\right) \times \exp\left(-\dfrac{Z'-1^{\ 2}}{2}\right)$, then set $X, Y, Z\ =\ X', Y', Z'$ and select X', Y', Z' ;otherwise reject X', Y', Z' and repeat the process from Step(1).

Conclusion

Selection-Rejection Methodology is valid for any dimension of random variable(continuous or discrete).In this method we approximate the target function to proposal function so that after a number of successive iterations the proposal function becomes almost equal to target function and proposal function is selected.

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