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Graceful Related Labeling and its Applications

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Abstract

The present paper is aimed to focus on various graph operations on semi smooth graceful labeling to generate new families of graceful related labeling. In this paper we proved that every semi smooth graceful graph is (k, d)-graceful, k-graceful and odd-even graceful. We also proved that a graph obtained by joining two semi smooth graceful graphs G, H by a path P_n of arbitrary length is graceful. We got equivalence of graceful labeling and odd-even graceful labeling too.

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1. Introduction

In 1966 A. Rosa [7] defined α-labeling as a graceful labeling f with an additional property that there is an integer $t \in \{0, 1, \ldots, q-1\}$ such that for every edge $e = (u, v)$, either $f(u) \leq t < f(v)$ or $f(v) \leq t < f(u)$. A graph which admits α-labeling is necessarily bipartite graph with partition $V_1 = \{w \in V(G)/f(w) \leq t\}$, $V_2 = \{w \in V(G)/f(w) > t\}$. A natural generalization of graceful labeling is the notion of k-graceful labeling and (k, d)-graceful labeling. Obviously 1-graceful or $(1, 1)$-graceful graph is graceful. A graph which admits α-labeling is always k-graceful graph, $\forall\ k \in N$. H. K. Ng [6] has identified some graphs that are k-graceful, $\forall\ k \in N$, but do not have α-labeling. Particularly cycle C_{4t+2} $(t \in N)$ is (1, 2)-graceful graph, but it has no α-labeling.

V. J. Kaneria and M. M. Jariya [4] define smooth graceful labeling and they proved cycle C_n $(n \equiv 0 \pmod 4)$, path P_n, grid graph $P_n \times P_m$ and complete bipartite graph $K_{2,n}$ are smooth graceful graphs.

We will consider a simple undirected finite graph $G = (V, E)$ on $|V| = p$ vertices and $|E| = q$ edges. For all terminology and standard notations we follows Harary [3] For a comprehensive bibliography of papers on different graph labeling are given by Gallian [2]. Here we will recall some definitions which are used in this paper.

Definition 1.1. A function f is called *graceful labeling* of a graph $G = (V, E)$ if $f : V(G) \longrightarrow \{0, 1, \ldots, q\}$ is injective and the induced function $f^\star : E(G) \longrightarrow \{1, 2, \ldots, q\}$ defined as $f^\star(e) = |f(u) - f(v)|$ is bijective for every edge $e = (u, v) \in E(G)$. A graph G is called *graceful graph* if it admits a graceful labeling.

Definition 1.2. A function f is called (k, d)*-graceful labeling* of a graph $G = (V, E)$ if $f : V(G) \longrightarrow \{0, 1, \ldots, k + (q - 1)d\}$ is injective and the induced function $f^\star : E(G) \longrightarrow \{k, k + d, k + 2d, \ldots, k + (q - 1)d\}$ defined as $f^\star(e) = |f(u) - f(v)|$ is bijective for every edge $e = (u, v) \in E(G)$. A graph G is called (k, d)*-graceful graph* if it admits a (k, d)-graceful labeling.

Above definition was introduced by Acharya and Hegde [1].

Definition 1.3. A function f is called k*-graceful labeling* of a graph $G = (V, E)$ if $f : V(G) \longrightarrow \{0, 1, \ldots, k+q-1\}$ is injective and the induced function $f^\star : E(G) \longrightarrow \{k, k+1, k+2, \ldots, k+q-1\}$ defined as $f^\star(e) = |f(u) - f(v)|$ is bijective for every edge $e = (u, v) \in E(G)$. A graph G is called k*-graceful graph* if it admits a k-graceful labeling.

Obviously $(k, 1)$-graceful labeling and $(1, 1)$-graceful labeling are k-graceful labeling and graceful labeling respectively.

Definition 1.4. A function f is called *odd-even graceful labeling* of a graph $G = (V, E)$ if $f : V(G) \longrightarrow \{1, 3, 5, \ldots, 2q + 1\}$ is injective and the induced function $f^\star : E(G) \longrightarrow \{2, 4, \ldots, 2q\}$ defined as $f^\star(e) = |f(u) - f(v)|$ is bijective for every edge $e = (u, v) \in E(G)$. A graph G is called *odd-even graceful graph* if it admits an odd-even graceful labeling.

Above definition was introduced by Shridevi et al. [8]. In same paper they proved that path P_n, star $K_{1,n}$, complete bipartite graph $K_{m,n}$ and bistar $B_{m,n}$ are odd-even graceful.

Definition 1.5. A smooth graceful graph G, we mean it is a bipartite graph with $|E(G)| = q$ and the property that for all non-negative integer l, there is a $1 - 1$ function g :

$V(G) \longrightarrow \{0, 1, \ldots, \left\lfloor \frac{q-1}{2} \right\rfloor, \left\lfloor \frac{q+1}{2} \right\rfloor + l, \left\lfloor \frac{q+3}{2} \right\rfloor + l, \ldots, q + l\}$ such that the induced edge labeling function $g^\star : E(G) \longrightarrow \{1 + l, 2 + l, \ldots, q + l\}$ defined as $f^\star(e) = |f(u) - f(v)|$ is a bijection for every edge $e = (u, v) \in E(G)$.

Definition 1.6. A semi smooth graceful graph G, we mean it is a bipartite graph with $|E(G)| = q$ and the property that for all non-negative integer l, there is an integer t $(0 < t \leq q)$ and an injective function $g : V(G) \longrightarrow \{0, 1, \ldots, t-1, t+l, t+l+1, \ldots, q+l\}$ such that the induced edge labeling function $g^\star : E(G) \longrightarrow \{1 + l, 2 + l, \ldots, q + l\}$ defined as $f^\star(e) = |f(u) - f(v)|$ is a bijection for every edge $e = (u, v) \in E(G)$.

If we take $l = 0$ in above both definitions 1.5, 1.6 the labeling functions g will become graceful labeling for the graph G. Every smooth graceful graph is also a semi smooth graceful graph by taking $t = \left\lfloor \frac{q+1}{2} \right\rfloor$.

2. Main Results

Theorem 2.1. Every semi smooth graceful graph is also (k, d)-graceful.

Proof. Let G be a semi smooth graceful graph with semi smooth vertex labeling function $g : V(G) \longrightarrow \{0, 1, \ldots, t-1, t+l, t+l+1, \ldots, q+l\}$, whose induced edge labeling function $g^\star : E(G) \longrightarrow \{l + 1, l + 2, \ldots, q + l\}$ defined by $g^\star(e) = |g(u) - g(v)|$, $\forall e = (u, v) \in E(G)$, for some $t \in \{1, 2, \ldots, q\}$ and arbitrary non-negative integer l.

Since G is a bipartite graph, we will take $V(G) = V_1 \cup V_2$ (where $V_1 \neq \phi$, $V_2 \neq \phi$ and $V_1 \cap V_2 = \phi$) and there is no edge $e \in E(G)$ whose both end vertices lies in V_1 or V_2. Moreover

$$\{g(u)/u \in V_1\} \subset \{1, 2, \ldots, t - 1\}$$

$$\{g(u)/u \in V_2\} \subset \{t + l, t + l + 1, \ldots, q + l\}.$$

Otherwise by taking l sufficiently large, the induced edge function $g^\star$ produce edge label which is less than l, gives a contradiction that G admits a semi smooth graceful labeling g.

Now define $h : V(G) \longrightarrow \{0, 1, 2, \ldots, k + (q - 1)d\}$ as follows

$$h(u) = d \cdot g(u), \forall u \in V_1 \quad \text{and}$$
$$h(w) = d \cdot (g(w) - l) + (k - d), \forall w \in V_2.$$

Above labeling function h give rise (k, d)-graceful labeling to the graph G. Because for any edge $e = (w, u) \in E(G)$, we shall have $g^\star(e) = i + l$, for some $i \in \{1, 2, \ldots, q\}$ and l is an arbitrary non-negative integer.

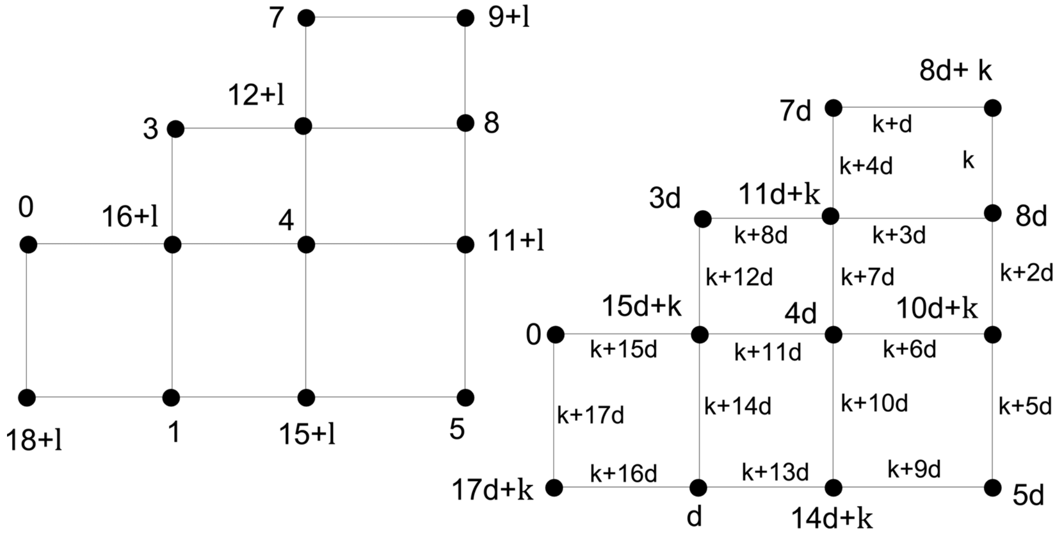

Figure 1: Semi smooth graceful labeling and (k, d)-graceful labeling for St_4.

Moreover $h^\star(e) = h(w) - h(u)$ (assuming $e = (w, u)$ with $u \in V_1$, $w \in V_2$)

$$\begin{aligned}
&= d(g(w) - l) + (k - d) - dg(u) \\
&= d(g(w) - g(u)) + k - (l + 1)d \\
&= d|g(w) - g(u)| + k - (l + 1)d \\
&= dg^\star(e) + k - (l + 1)d \\
&= d(i + l) - (l + 1)d + k \\
&= (i - 1)d + k \in Imh^\star \\
&= \{k, k + d, k + 2d, \dots, k + (q - 1)d\}.
\end{aligned}$$

Thus G admits a (k, d)-graceful labeling h and so it is a (k, d)-graceful graph.

Illustration 2.2. Semi smooth graceful labeling and (k, d)-graceful labeling for St_4 are shown in *figure*-1.

Corollary 2.3. Every semi smooth graceful graph is also k-graceful graph.

Theorem 2.4. A graph obtained by joining two semi smooth graceful graphs G, H with an arbitrary path P_n (path on n vertices) is a semi smooth graceful graph.

Proof. Let K be the graph obtained by joining two semi smooth graceful graphs G, H with an arbitrary path P_n. Let $q_1 = |E(G)|$, $q_3 = |E(H)|$. Let G, H be two semi smooth graceful graphs with semi smooth vertex labeling functions $g : V(G) \longrightarrow \{0, 1, \dots, t_1 - 1, t_1 + l_1, t_1 + l_1 + 1, \dots, q_1 + l_1\}$, $h : V(H) \longrightarrow \{0, 1, \dots, t_2 - 1, t_2 + l_2, \dots, q_3 + l_2\}$, whose induced edge labeling functions are absolute difference of end vertices for each edges and $t_1 \in \{1, 2, \dots, q_1\}$, $t_2 \in \{1, 2, \dots, q_3\}$, l_1, l_2 are arbitrary non-negative integers.

Since G, H both are bipartite graphs, we will have partitions of vertices sets as follows.

$$\begin{aligned}
&\{g(u)/u \in V_1\} \subset \{1, 2, \ldots, t_1 - 1\},\\
&\{g(u)/u \in V_2\} \subset \{t_1 + l_1, t_1 + l_1 + 1, \ldots, q_1 + l_1\},\\
&\{h(w)/w \in V_3\} \subset \{1, 2, \ldots, t_2 - 1\},\\
&\{h(w)/w \in V_4\} \subset \{t_2 + l_2, t_2 + l_2 + 1, \ldots, q_3 + l_2\} \text{ and}\\
&V(G) = V_1 \cup V_2,\ V(H) = V_3 \cup V_4.
\end{aligned}$$

Let $v_t \in V_1$ be such that $g(v_t) = t_1 - 1$. Such vertex would be lies in V_1, otherwise $1 + l_1$ edge label can not be produce in G. In fact $1 + l_1$ edge label can be produce by the vertices of G whose vertex labels are $t_1 - 1$ and $t_1 + l_1$ and they are adjacent in G. Let $w_q \in V_3$ be such that $h(w_q) = 0$.

Now join v_t and w_q vertices of G and H respectively by an arbitrary path P_n of length $n - 1$, to produce the graph K [obtained by G, H and an arbitrary path P_n]. Let $w_1 = v_t, w_2, \ldots, w_n = w_q$ be consecutive vertices of the path P_n.

Now define

$$f : V(K) \longrightarrow \{0, 1, 2, \ldots, t - 1, t + l, t + l + 1, \ldots, q_1 + q_2 + q_3 + l\}$$

(where $q_2 = |E(P_n)|, t = t_1 + t_2 - 1 + \frac{q_2}{2}$, when q_2 is even or $t = t_1 - t_2 + q_3 + \frac{q_2 + 1}{2}$, when q_2 is odd and l be any arbitrary non-negative integer) as follows.

$$\begin{aligned}
f(u) &= g(u), \forall u \in V_1\\
&= g(u) + l - l_1 + (q_2 + q_3), \forall u \in V_2;\\
f(w_i) &= (q_2 + q_3) - \frac{i}{2} + t_1 + l, && \text{when } i \text{ is even}\\
&= t_1 - 1 + \frac{i - 1}{2}, && \text{when } i \text{ is odd}, \quad \forall i = 1, 2, \ldots, n;\\
f(w) &= t_1 + \frac{q_2 - 1}{2} + q_3 - h(w) + l, && \text{when } w \in V_3 \text{ and } q_2 \text{ is odd}\\
&= t_1 + \frac{q_2 - 1}{2} + q_3 + l_2 - h(w), && \text{when } w \in V_4 \text{ and } q_2 \text{ is odd}\\
&= t_1 + \frac{q_2 - 2}{2} + q_3 + h(w), && \text{when } w \in V_3 \text{ and } q_2 \text{ is even}\\
&= t_1 + l - l_2 + \frac{q_2 - 2}{2} + h(w), && \text{when } w \in V_4 \text{ and } q_2 \text{ is even.}
\end{aligned}$$

Above labeling function give rise semi smooth graceful labeling to the graph K and so it is a semi smooth graceful graph. ■

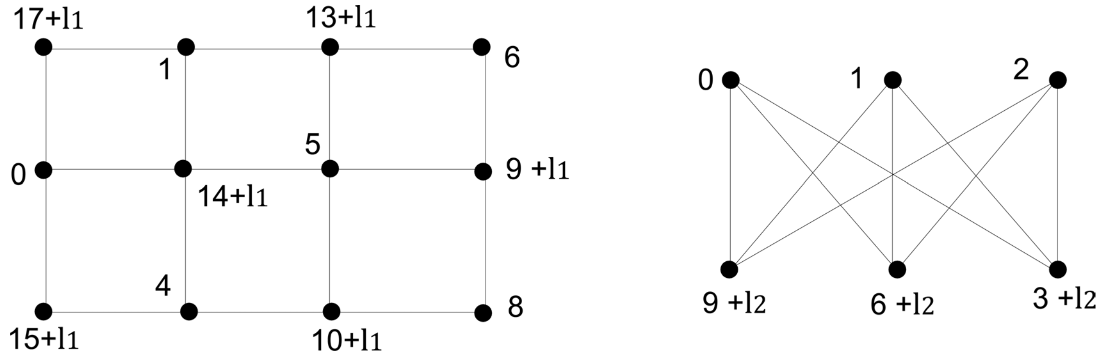

Figure 2: Semi smooth graceful labeling for $P_3 \times P_4$, $K_{3,3}$.

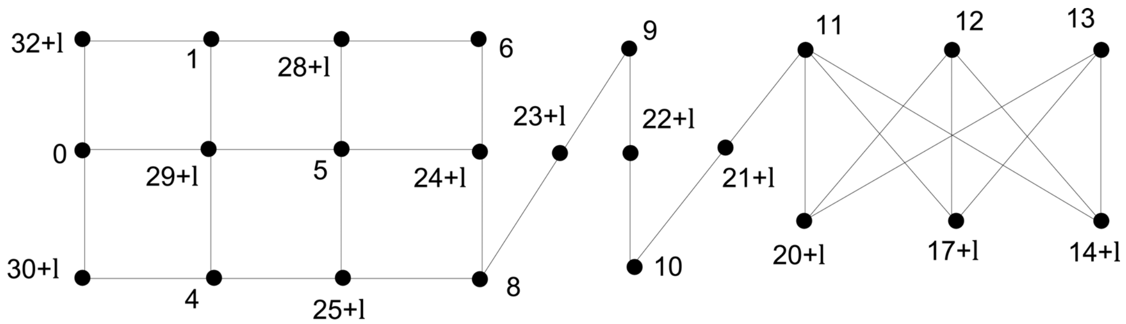

Figure 3: Semi smooth graceful labeling for a graph obtained by joining $P_3 \times P_4$ and $K_{3,3}$ with a path P_7.

Illustration 2.5. Semi smooth graceful labeling for $P_3 \times P_4$, $K_{3,3}$ and the graph obtained by joining $P_3 \times P_4$ and $K_{3,3}$ with a path P_7 (path on 7 vertices) are shown in *figure*-2 and 3.

Theorem 2.6. Every graceful graph is also an odd-even graceful graph.

Proof. Let G be a graceful graph with vertex graceful labeling function $f : V(G) \longrightarrow \{0, 1, \ldots, q\}$ whose induced edge labeling function is $f^\star : E(G) \longrightarrow \{1, 2, \ldots, q\}$ defined by $f^\star(e) = |f(u) - f(v)|, \forall e = (u, v) \in E(G)$ is a bijection.

Now define $h : V(G) \longrightarrow \{1, 3, 5, \ldots, 2q + 1\}$ as follows.

$$h(u) = 2f(u) + 1, \quad \forall u \in V(G).$$

Above labeling function h give rise an odd-even graceful labeling to the graph G.

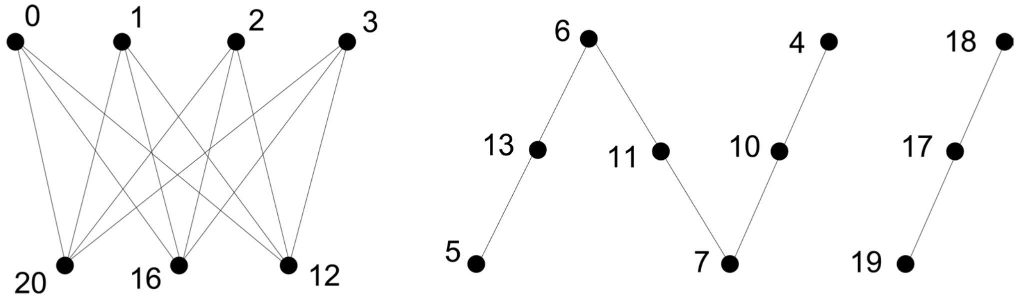

Figure 4: Graceful labeling for $K_{4,3} \cup P_7 \cup P_3$.

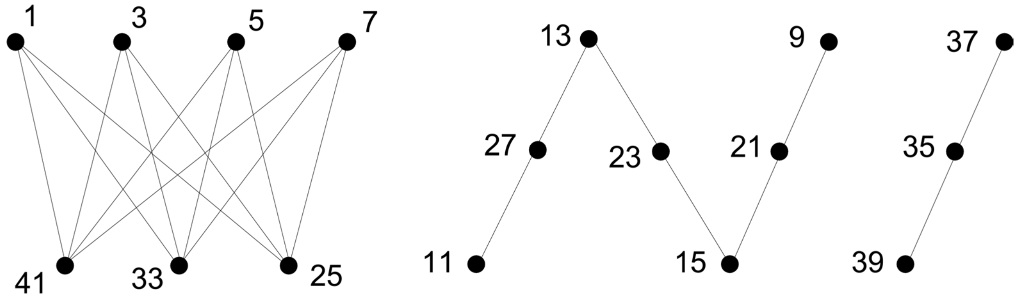

Figure 5: odd-even graceful labeling for $K_{4,3} \cup P_7 \cup P_3$.

Because for any edge $e = (x, y) \in E(G)$, we see that

$$\begin{aligned} h^\star(e) &= |h(x) - h(y)| \\ &= |2f(x) + 1 - (2f(y) + 1)| \\ &= |2(f(x) - f(y))| \\ &= 2|f(x) - f(y)| \\ &= 2|f^\star(e)| \end{aligned}$$

Since $f^\star$ is a bijection, so is $h^\star$. Thus G admits an odd-even graceful labeling h. Hence it is an odd-even graceful graph. ■

Illustration 2.7. Graceful labeling and odd-even graceful labeling for a graph $G = K_{4,3} \cup P_7 \cup P_3$ are shown in *figure*−4 and 5 respectively.

Corollary 2.8. Every semi smooth graceful graph is odd-even graceful graph.

Theorem 2.9. Every odd-even graceful graph is graceful.

Proof. Let G be an odd-even graceful graph with vertex labeling function $f : V(G) \longrightarrow \{1, 3, \ldots, 2q + 1\}$, where $q = |E(G)|$. Its induced edge labeling function is $f^\star : E(G) \longrightarrow \{2, 4, \ldots, 2q\}$ defined by $f^\star(e) = |f(u) - f(v)|$, $\forall e = (u, v) \in E(G)$. It is obvious that f is injective and $f^\star$ is bijective.

Now define $g : V(G) \longrightarrow \{0, 1, \dots, q\}$ as follows.

$$g(u) = \frac{f(u) - 1}{2}, \quad \forall u \in V(G).$$

Above labeling function g is also an injective function and $g^\star : E(G) \longrightarrow \{1, 2, \dots, q\}$ defined as $g^\star(e) = |g(u) - g(v)|$, $\forall e = (u, v) \in E(G)$ is a bijection. Because for any edge $e = (x, y) \in E(G)$,

$$\begin{aligned} g^\star(e) &= |g(x) - g(y)| \\ &= \left|\left(\frac{f(x) - 1}{2}\right) - \left(\frac{f(y) - 1}{2}\right)\right| \\ &= \frac{1}{2}|(f(x) - f(y))| \\ &= \frac{1}{2}|f^\star(e)| \text{ and } g^\star \text{ is a bijective function.} \end{aligned}$$

Thus G admits a graceful labeling g and hence it is a graceful graph. ■

3. Concluding Remarks

We discussed here graceful labeling and odd-even graceful labeling are equivalent. We also discussed about some graceful related graphs.

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