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2-primal Semiring

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Abstract

In this paper we introduce the notion of 2-primal semiring similar to the notion in ring. We also give some characterizations of 2-primal semirings by using prime ideals and insertion of factors property.

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1. Introduction

The study of 2-primal rings was inaugurated by G. Shin [5] (although the name “2-primal" was not coined at that time). Also he proved that a ring is 2-primal if and only if each of its minimal prime ideals is completely prime. The name “2-primal" was first introduced by Birkenmeier-Heatherly-Lee in [4]. Essential properties of 2-primal rings are developed in [1], [2] and [7].

In ths paper we introduce the concept of 2-primal semirings. For a prime ideal P of a semiring S, we define the subsets $O(P)$, O_P, $N(P)$, N_P, $\underline{O(P)}$, $\underline{N(P)}$, $\underline{O}_P$ and $\underline{N}_P$ of S as in ring and using these subsets we characterize 2-primal semirings. Also we generalise many results of 2-primal rings in 2-primal semirings. Some earlier works on semirings of the author may be found in [9], [10], [11] and [12].

2. Preliminaries

Definition 2.1. A nonempty set S is said to form a semiring with respect to two binary compositions, addition $(+)$ and multiplication $(\cdot)$ defined on it, if the following conditions are satisfied.

(1) $(S,+)$ is a commutative semigroup with zero,

(2) $(S,\cdot)$ is a semigroup,

(3) for any three elements $a,b,c \in S$
the left distributive law $a\cdot(b+c) = a\cdot b + a\cdot c$ and
the right distributive law $(b+c)\cdot a = b\cdot a + c\cdot a$ both hold and

(4) $s\cdot 0 = 0\cdot s = 0$ for all $s \in S$.

If S contains the multiplicative identity 1, then S is called a semiring with identity. Throughout this paper we assume a semiring S means a semiring with identity.

Definition 2.2. A nonempty subset I of a semiring S is called a left ideal of S if (i) $a,b \in I$ implies $a+b \in I$ and (ii) $a \in I$, $s \in S$ implies $s.a \in I$.

Similarly we can define right ideal of a semiring. A nonempty subset I of a semiring S is an ideal if it is a left ideal as well as a right ideal of S.

Definition 2.3. [3] An ideal I of a semiring S is called a k-ideal if $a+b \in I$ and $a \in I$ implies $b \in I$.

Definition 2.4. [6] A proper ideal I of a semiring S is called a prime ideal if $AB \subseteq I$ implies either $A \subseteq I$ or $B \subseteq I$, where A and B are ideals of S.

Definition 2.5. [6] A proper ideal I of a semiring S is called a semiprime ideal if $A^2 \subseteq I$ implies $A \subseteq I$, where A is an ideal of S.

Definition 2.6. A semiring S is called a prime semiring if $\{0\}$ is a prime ideal of S.

Definition 2.7. A semiring S is called a semiprime semiring if $\{0\}$ is a semiprime ideal of S.

Definition 2.8. An ideal I of a semiring S is said to be completely prime if $ab \in I$ implies $a \in I$ or $b \in I$ for $a,b \in S$.

Definition 2.9. An ideal I of a semiring S is said to be completely semiprime if $a^2 \in I$ implies $a \in I$ for $a \in S$.

Definition 2.10. A subset M of a semiring S is said to be m-system if for any $a,b \in M$, there exists $s \in S$ such that $asb \in M$.

Definition 2.11. [6] Let I be a proper ideal of a semiring S. Then the congruence on S, denoted by ρ_I and defined by $s\rho_I s'$ if and only if $s+a_1 = s'+a_2$ for some $a_1,a_2 \in I$, is called the Bourne congruence on S defined by the ideal I.

We denote the Bourne congruence (ρ_I) class of an element r of S by r/ρ_I or simply by r/I and denote the set of all such congruence classes of S by S/ρ_I or simply by S/I.

It should be noted that for any $s \in S$ and for any proper ideal I of S, s/I is not necessarily equal to $s + I = \{s + a : a \in I\}$ but surely contains it.

Definition 2.12. [6] For any proper ideal I of S if the Bourne congruence ρ_I, defined by I, is proper i.e. $0/I \neq S$ then we define the addition and multiplication on S/I by $a/I + b/I = (a + b)/I$ and $(a/I)(b/I) = (ab)/I$ for all $a, b \in S$. With these two operations S/I forms a semiring and is called the Bourne factor semiring or simply the factor semiring.

Definition 2.13. Let A be a non-empty subset of a semiring S. Right annihilator of A in S, denoted by $ann_R(A)$, is defined by $ann_R(A) = \{s \in S : As = 0\}$.

If $A = \{a\}$, then we denote $ann_R(A)$ by $ann_R(a)$.

Analogously we can define left annihilator ($ann_L(A)$) of A. Annihilator of a set A is denoted by $ann(A)$ which is left as well as right annihilator of A.

Remark 2.14. If S is a semiring with absorbing zero then $ann_R(A)$ is a right ideal of S and $ann_L(A)$ is a left ideal of S. If A is an ideal of S then both annihilators are ideals of S.

3. 2-primal semiring

Definition 3.1. A semiring S is said to be 2-primal semiring if $\mathcal{P}(S) = \mathcal{N}(S)$, where $\mathcal{P}(S)$ denotes the prime radical of S i.e. intersection of all prime ideals of S and $\mathcal{N}(S)$ denotes the set of all nilpotent elements of S.

Definition 3.2. A semiring S is said to be reduced if it has no nonzero nilpotent elements.

Proposition 3.3. Every reduced semiring is 2-primal.

Proof. Since for any semiring S, $\mathcal{P}(S) \subseteq \mathcal{N}(S)$, reduced semirings are 2-primal. ■

Definition 3.4. An ideal I of a semiring S is said to have the insertion of factors property or simply IFP if $ab \in I$ implies $aSb \subseteq I$ for $a, b \in S$.

Definition 3.5. An ideal I of a semiring S is said to be right (left) symmetric if $abc \in I$ implies $acb \in I$ (respectively $bac \in I$) for $a, b, c \in S$.

Definition 3.6. A semiring S is said to be satisfy (SI) if for each $a \in S$, $ann_R(a)$ is an ideal of S.

Lemma 3.7. For any semiring S the following statements are equivalent:

(i) S satisfies (SI).

(ii) For any $a, b \in S$, $ab = 0$ implies $aSb = 0$.

Proof. $(i) \Rightarrow (ii)$ Let $ab = 0$, for $a, b \in S$. Then $b \in ann_R(a)$. As $ann_R(a)$ is an ideal of S, $Sb \subseteq ann_R(a)$. So $aSb = 0$.

$(ii) \Rightarrow (i)$ Obviously $ann_R(a)$ is a right ideal of S for each $a \in S$. Let $b \in ann_R(a)$ and $s \in S$. Then $ab = 0$ and by (ii), $aSb = 0$. So $S(ann_R(a)) \subseteq ann_R(a)$. Therefore $ann_R(a)$ is an ideal of S for each $a \in S$. ∎

Proposition 3.8. If S satisfies (SI), then S is a 2-primal semiring.

Proof. We know $\mathcal{P}(S) \subseteq \mathcal{N}(S)$. Suppose $a \in \mathcal{N}(S)$, then $a^n = 0$, for some positive integer n. If possible let $a \notin \mathcal{P}(S)$. Then $a \notin P$ for some prime ideal P of S i.e. $a \in S - P$. As P is prime, $S - P$ is an m-system. So there exists $s_1 \in S$ such that $as_1a \in S - P$. Again since as_1a, $a \in S - P$, there exists $s_2 \in S$ such that $as_1as_2a \in S - P$. Continuing this process, there exist $s_3, s_4, \dots, s_{n-1}$ in S such that $as_1as_2a, \dots, as_{n-1}a \in S - P$. Since S satisfies (SI), by Lemma 3.7, $a^n = 0$ i.e. $aa^{n-1} = 0$ implies $as_1a^{n-1} = 0 \Rightarrow (as_1a)a^{n-2} = 0 \Rightarrow (as_1a)s_2a^{n-2} = 0$ [by Lemma 3.7]. Continuing this process, we get $as_1as_2a \dots as_{n-1}a = 0 \in P$, a contradiction. Thus $a \in \mathcal{P}(S)$. So $\mathcal{P}(S) = \mathcal{N}(S)$ i.e. S is a 2-primal semiring. ∎

Definition 3.9. For a prime ideal P of a semiring S, we define
$O(P) = \{x \in S : xSy = 0 \text{ for some } y \in S - P\}$.
$\overline{O(P)} = \{x \in S : x^n \in O(P) \text{ for some positive integer } n\}$.

$O_P = \{x \in S : xy = 0 \text{ for some } y \in S - P\}$.
$\overline{O}_P = \{x \in S : x^n \in O_P \text{ for some positive integer } n\}$.
$N(P) = \{x \in S : xSy \subseteq \mathcal{P}(S) \text{ for some } y \in S - P\}$.
$\overline{N(P)} = \{x \in S : x^n \in N(P) \text{ for some positive integer } n\}$.
$N_P = \{x \in S : xy \in \mathcal{P}(S) \text{ for some } y \in S - P\}$.
$\overline{N}_P = \{x \in S : x^n \in N_P \text{ for some positive integer } n\}$.

Now $O(P)$ and $N(P)$ are subsets of P, $O(P) \subseteq O_P \subseteq \overline{O}_P$ and $N(P) \subseteq N_P \subseteq \overline{N}_P$ for each prime ideal P of S.

Proposition 3.10. Let S be a semiring and P be a prime ideal of S. Then
$O(P) = \{x \in S : xS < y >= 0 \text{ for some } y \in S - P\}$ and
$N(P) = \{x \in S : xS < y > \subseteq \mathcal{P}(S) \text{ for some } y \in S - P\}$, where $< y >$ denotes the ideal of S generated by y.

Proof. Let $A = \{x \in S : xS < y >= 0 \text{ for some } y \in S - P\}$. Clearly $A \subseteq O(P)$. Suppose $x \in O(P)$. Then $xSy = 0$ for some $y \in S - P$. Now elements of $< y >$ are of the form $s'y + ys'' + ny + \sum_{i=1}^{m} s_i y s'_i$, where $s', s'', s_i, s'_i \in S$ and n is a non-negative integer. So $xS < y >= 0$. Therefore $x \in A$. Thus $O(P) = A$.

As $\mathcal{P}(S)$ is an ideal of S, the proof of the second part is similar as first part. ∎

Proposition 3.11. Let S be a semiring and P be a prime ideal of S. Then $O(P)$ and

$N(P)$ are k-ideals of S.

Proof. $O(P)$ is a nonempty subset of S, since $0 \in O(P)$. Let $x_1, x_2 \in O(P)$. Then there exist y_1 and y_2 in $S-P$ such that $x_1S < y_1 >= 0$ and $x_2S < y_2 >= 0$. Since P is a prime ideal of S, $S-P$ is a m-system. So there exists $s \in S$ such that $y_1sy_2 \in S-P$. Now $< y_1sy_2 >\subseteq< y_1 >$ and $< y_1sy_2 >\subseteq< y_2 >$. Therefore $(x_1+x_2)S < y_1sy_2 >= 0$. Thus $x_1+x_2 \in O(P)$.

Let $x \in O(P)$. Then there exists $y \in S-P$ such that $xS < y >= 0$. Therefore $SxS < y >= 0$ and $xSS < y >\subseteq xS < y >= 0$. Thus $Sx, xS \subseteq O(P)$. So $O(P)$ is an ideal of S.

Let $x_1+x_2 \in O(P)$ and $x_1 \in O(P)$. Then there exist y_1 and y_2 in $S-P$ such that $(x_1+x_2)S < y_1 >= 0$ and $x_1S < y_2 >= 0$. Since $S-P$ is an m-system, there exists s such that $y_1sy_2 \in S-P$. Now $< y_1sy_2 >\subseteq< y_1 >$ and $< y_1sy_2 >\subseteq< y_2 >$. So $(x_1+x_2)S < y_1sy_2 >= 0$ and $x_1S < y_1sy_2 >= 0$. Therefore $x_2S < y_1sy_2 >= 0$. Thus $x_2 \in O(P)$. Hence $O(P)$ is a k-ideal of S.

Since $\mathcal{P}(S)$ is a k-ideal of S, by similar argument we can prove that $N(P)$ is a k-ideal of S. ∎

Proposition 3.12. Let S be a semiring and P be a prime ideal of S such that O_P and N_P are ideals of S.

(i) If O_P (resp. N_P) has the IFP, then $\overline{O}_P$ (resp. $\overline{N}_P$) is an ideal of S.

(ii) O_P (resp. N_P) is a completely semiprime ideal of S if and only if $O_P = \overline{O}_P$ (resp. $N_P = \overline{N}_P$).

Proof.

(i) Clearly $\overline{O}_P$ is a nonempty subset of S. Let $x, y \in \overline{O}_P$. Then $x^n, y^m \in O_P$, for some positive integers n, m. Since O_P has the IFP, the elements of the form $xs_1xs_2x\ldots xs_{k-1}x$ $(k \geq n)$ and $ys_1ys_2y\ldots ys_{r-1}y$ $(r \geq m)$ belong to O_P i.e. an expression contains at least n x's or m y's must belongs to O_P. Now each term of $(x+y)^{m+n}$ contains at least n x's or m y's. Since O_P is an ideal $(x+y)^{m+n} \in O_P$. Also $(sx)^n, (xs)^n \in O_P$, for each $s \in S$ i.e. $x+y, sx, xs \in \overline{O}_P$, for each $s \in S$. Therefore $\overline{O}_P$ is an ideal of S. Similarly it can be proved that $\overline{N}_P$ is an ideal of S.

(ii) Suppose O_P is a completely semiprime ideal of S. Clearly $O_P \subseteq \overline{O}_P$. Let $a \in \overline{O}_P$. Then $a^n \in O_P$, for some positive integer n. As O_P is completely semiprime ideal of S, $a^n \in O_P$ implies $a \in O_P$. Therefore $O_P = \overline{O}_P$. The converse part is obvious. By the same method, N_P is a completely semiprime ideal of S if and only if $N_P = \overline{N}_P$. ∎

Proposition 3.13. Let S be a semiring. Then $\mathcal{N}(S) \subseteq \bigcap_{P\in Spec(S)} \overline{O}_P \subseteq \bigcap_{Q\in mSpec(S)} \overline{O}_Q$, where $Spec(S)$ and $mSpec(S)$ denote the set of all prime and minimal prime ideals of S respectively.

Proof. We first show that if P_1 and P_2 are two prime ideals of S such that $P_1 \subseteq P_2$, then $\overline{O}_{P_2} \subseteq \overline{O}_{P_1}$. Let $a \in \overline{O}_{P_2}$. Then $a^n \in O_{P_2}$, for some positive integer n, which implies that $a^n b = 0$, for some $b \in S - P_2$. i.e. $b \in S - P_1$. So $a^n \in O_{P_1}$. Thus $a \in \overline{O}_{P_1}$.

Let P be any prime ideal of S, then there exists a minimal prime ideal Q of S such that $Q \subseteq P$. Therefore $\bigcap\limits_{P \in Spec(S)} \overline{O}_P \subseteq \bigcap\limits_{Q \in mSpec(S)} \overline{O}_Q$.

Let $a \in \mathcal{N}(S)$. So $a^n = 0$, for some positive integer n. Therefore $a^n \in O_P$, for each prime ideal P of S i.e. $a \in \overline{O}_P$ for each prime ideal P of S, which implies that $a \in \bigcap\limits_{P \in Spec(S)} \overline{O}_P$. Hence $\mathcal{N}(S) \subseteq \bigcap\limits_{P \in Spec(S)} \overline{O}_P \subseteq \bigcap\limits_{Q \in mSpec(S)} \overline{O}_Q$. ■

Proposition 3.14. Let S be a semiring. Then $\mathcal{P}(S) = \bigcap\limits_{P \in Spec(S)} N(P) = \bigcap\limits_{Q \in mSpec(S)} N(Q)$.

Proof. Let $a \in \mathcal{P}(S)$. Then $aS \subseteq \mathcal{P}(S)$. Since $1 \notin P$ for any prime ideal P of S, $a \in N(P)$ for every prime ideal P of S i.e. $a \in \bigcap\limits_{P \in Spec(S)} N(P)$. So $\mathcal{P}(S) \subseteq \bigcap\limits_{P \in Spec(S)} N(P)$. Also $mSpec(S) \subseteq Spec(S)$ implies $\bigcap\limits_{P \in Spec(S)} N(P) \subseteq \bigcap\limits_{Q \in mSpec(S)} N(Q)$. Again $N(P) \subseteq P$ for any prime ideal P of S. So $\bigcap\limits_{Q \in mSpec(S)} N(Q) \subseteq \bigcap\limits_{Q \in mSpec(S)} Q = \mathcal{P}(S)$. Therefore $\mathcal{P}(S) = \bigcap\limits_{P \in Spec(S)} N(P) = \bigcap\limits_{Q \in mSpec(S)} N(Q)$. ■

Theorem 3.15. For a semiring S the following statements are equivalent:

(1) S is a 2-primal semiring.

(2) $\mathcal{P}(S)$ is a completely semiprime ideal of S.

(3) $\mathcal{P}(S)$ is a left and right symmetric ideal of S.

(4) $xy \in \mathcal{P}(S)$ implies $ySx \subseteq \mathcal{P}(S)$ for $x, y \in S$.

Proof. (1) $\Rightarrow$ (2) Let $a^2 \in \mathcal{P}(S)$, where $a \in S$. Then $a^2 \in \mathcal{N}(S)$ [since $\mathcal{P}(S) = \mathcal{N}(S)$] which implies that $(a^2)^n = 0$, for some positive integer n i.e. $a^{2n} = 0$. So $a \in \mathcal{N}(S) = \mathcal{P}(S)$. Therefore $\mathcal{P}(S)$ is a completely semiprime ideal of S.

(2) $\Rightarrow$ (3) Let $abc \in \mathcal{P}(S)$, where $a, b, c \in S$. Now $(cab)^2 = c(abc)ab \in \mathcal{P}(S)$. Since $\mathcal{P}(S)$ is completely semiprime, $cab \in \mathcal{P}(S)$. $(abac)^2 = aba(cab)ac \in \mathcal{P}(S) \Rightarrow abac \in \mathcal{P}(S) \Rightarrow (bacba)^2 = bacb(abac)ba \in \mathcal{P}(S) \Rightarrow bacba \in \mathcal{P}(S) \Rightarrow (acb)^3 = ac(bacba)cb \in \mathcal{P}(S) \Rightarrow acb \in \mathcal{P}(S)$. Also $(bac)^2 = b(acb)ac \in \mathcal{P}(S) \Rightarrow bac \in \mathcal{P}(S)$. Therefore $\mathcal{P}(S)$ is a left and right symmetric ideal of S.

(3) $\Rightarrow$ (4) Let $xy \in \mathcal{P}(S)$, where $x, y \in S$. Suppose $s \in S$, then $sxy \in \mathcal{P}(S)$. As $\mathcal{P}(S)$ is right symmetric $syx \in \mathcal{P}(S)$. Also since $\mathcal{P}(S)$ is left symmetric, $ysx \in \mathcal{P}(S)$. Therefore $ySx \subseteq \mathcal{P}(S)$.

(4) $\Rightarrow$ (1) We know $\mathcal{P}(S) \subseteq \mathcal{N}(S)$. Let $x \in \mathcal{N}(S)$, then $x^n = 0$, for some positive integer n. If possible, let $x \notin P$, for some prime ideal P of S. Then $x \in S - P$. As P is prime, $S - P$ is an m-system. So there exists s_1 in S such that $xs_1x \in S - P$. Continuing this process there exist $s_2, s_3, \ldots, s_{n-1} \in S$ such that $xs_1xs_2x \ldots xs_{n-1}x \in S - P$. Now by (4), $x^n \in \mathcal{P}(S)$ implies $xs_1xs_2x \ldots xs_{n-1}x \in \mathcal{P}(S)$ i.e. $xs_1xs_2x \ldots xs_{n-1}x \in P$, a contradiction. Thus $x \in \mathcal{P}(S)$. Therefore $\mathcal{P}(S) = \mathcal{N}(S)$. Hence S is a 2-primal semiring. ■

Theorem 3.16. The following statements are equivalent for a semiring S:

(i) S is a 2-primal semiring.

(ii) $\mathcal{P}(S)$ has the IFP.

(iii) $N(P)$ has the IFP for each prime ideal P of S.

(iv) $N(P) = \overline{N}_P$ for each prime ideal P of S.

(v) $N(P) = N_P$ for each prime ideal P of S.

(vi) $N_P \subseteq P$ for each prime ideal P of S.

(vii) $N_{P/\mathcal{P}(S)} \subseteq P/\mathcal{P}(S)$ for each prime ideal P of S.

Proof. $(i) \Rightarrow (ii)$ Let S be a 2-primal semiring. Let $xy \in \mathcal{P}(S)$ and $s \in S$. Then $sxy \in \mathcal{P}(S)$. Now by Theorem 3.15(3), $\mathcal{P}(S)$ is a left symmetric ideal of S. So $xsy \in \mathcal{P}(S)$. Thus $xSy \subseteq \mathcal{P}(S)$ i.e. $\mathcal{P}(S)$ has the IFP.

$(ii) \Rightarrow (iii)$ Let $xy \in N(P)$, where P is a prime ideal of S. So $xySb \subseteq \mathcal{P}(S)$ for some $b \in S - P$. Since $\mathcal{P}(S)$ has the IFP, $xSySb \subseteq \mathcal{P}(S)$. Therefore $xSy \subseteq N(P)$. Thus $N(P)$ has the IFP for each prime ideal P of S.

$(iii) \Rightarrow (i)$ Always $\mathcal{P}(S) \subseteq \mathcal{N}(S)$. Let $a \in \mathcal{N}(S)$. Then $a^n = 0$, for some positive integer n. If possible suppose $a \notin \mathcal{P}(S)$, then there exists a prime ideal P of S such that $a \notin P$. As P is prime ideal of S, $S - P$ is an m-system of S. So there exists $s_1 \in S$ such that $as_1a \notin P$. Continuing this process we get $s_2, s_3, \ldots, s_{n-1} \in S$ such that $as_1as_2a \ldots as_{n-1}a \notin P$. Also since $N(P)$ has the IFP, $a^n = 0 \in N(P) \Rightarrow as_1as_2a \ldots as_{n-1}a \in N(P)$. As $N(P) \subseteq P$, $as_1as_2a \ldots as_{n-1}a \in P$, a contradiction. So $a \in \mathcal{P}(S)$. Hence $\mathcal{P}(S) = \mathcal{N}(S)$ i.e. S is a 2-primal semiring.

$(i) \Rightarrow (iv)$ Let P be a prime ideal of S and $x \in N(P)$. Then there exists $y \in S - P$ such that $xSy \subseteq \mathcal{P}(S)$. Since S contains the identity element $xy \in \mathcal{P}(S)$ i.e. $x \in N_P \subseteq \overline{N}_P$.

So $N(P) \subseteq \overline{N}_P$. Conversely, let $a \in \overline{N}_P$. Then $a^n \in N_P$, for some positive integer n. So there exists $b \in S - P$ such that $a^n b \subseteq \mathcal{P}(S)$ i.e. $a^{n-1}(ab) \in \mathcal{P}(S)$. Therefore $a^{n-1}(ab)b \in \mathcal{P}(S)$, as $\mathcal{P}(S)$ is an ideal of S.

$\Rightarrow a^{n-1}b(ab)\mathcal{P}(S)$, as $\mathcal{P}(S)$ is right symmetric by Theorem 3.15(3),
$\Rightarrow a^{n-2}(ab)^2\mathcal{P}(S)$.
Continuing this process we get $(ab)^n \in \mathcal{P}(S)$. Since S is a 2-primal semiring, by Theorem 3.15(2), $\mathcal{P}(S)$ is a completely semiprime semiring. So $ab \in \mathcal{P}(S)$. Now by (ii), $\mathcal{P}(S)$ has the IFP. Therefore $aSb \subseteq \mathcal{P}(S)$ which implies that $a \in N(P)$. So $\overline{N}_P \subseteq N(P)$. Hence $N(P) = \overline{N}_P$ for each prime ideal P of S.

$(iv) \Rightarrow (v)$ Follows from the fact $N(P) \subseteq N_P \subseteq \overline{N}_P$.

$(v) \Rightarrow (vi)$ As $N(P) \subseteq P$.

$(vi) \Rightarrow (vii)$ Suppose $\overline{S} = S/\mathcal{P}(S)$ and $\overline{P} = P/\mathcal{P}(S)$ for every prime ideal P of S. Let $\overline{a} \in N_{\overline{P}}$. So there exists $\overline{b} \in \overline{S} - \overline{P}$ such that $\overline{a}\overline{b} \in \overline{\mathcal{P}(S)}$ i.e. $(a/\mathcal{P}(S))(b/\mathcal{P}(S)) = (ab)/\mathcal{P}(S) = 0/\mathcal{P}(S)$. Since $\mathcal{P}(S)$ is a k-ideal of S, $ab \in \mathcal{P}(S)$. So $a \in N_P$. As $N_P \subseteq P$, $a \in P$ i.e. $\overline{a} \in \overline{P}$. Thus $N_{\overline{P}} \subseteq \overline{P}$.

$(vii) \Rightarrow (i)$ We first prove that if $S/\mathcal{P}(S)$ is reduced, then S is a 2-primal semiring. Now $\mathcal{P}(S) \subseteq \mathcal{N}(S)$. To prove the reverse inclusion, let $a \in \mathcal{N}(S)$. Then $a^n = 0$, for some positive integer n. So $a/\mathcal{P}(S)$ is a nilpotent element of $S/\mathcal{P}(S)$. Since $S/\mathcal{P}(S)$ is reduced, it has no nonzero nilpotent element. So $a/\mathcal{P}(S) = 0/\mathcal{P}(S)$. As $\mathcal{P}(S)$ is a k-ideal of S, $a \in \mathcal{P}(S)$ i.e. $\mathcal{N}(S) \subseteq \mathcal{P}(S)$.

We now prove that $\overline{S} = S/\mathcal{P}(S)$ is a reduced semiring. If possible let $\overline{S}$ be not a reduced semiring. Then there exists a nonzero element $\overline{a} \in \overline{S}$ such that $\overline{a}^2 = \overline{0}$. Since $\overline{a} \neq \overline{0}$, $a \notin \mathcal{P}(S)$. So there exists a prime ideal P of S such that $a \notin P$. Thus $\overline{a} \notin \overline{P}$ i.e. $\overline{a} \in \overline{S} - \overline{P}$ and $\overline{a}^2 = \overline{0}$, which implies that $\overline{a} \in N_{\overline{P}} \subseteq \overline{P}$, a contradiction. Therefore S is a reduced semiring and hence S is a 2-primal semiring. ■

Theorem 3.17. The following statements are equivalent for a semiring S:

(i) S is a 2-primal semiring.

(ii) $N(P)$ is a completely semiprime ideal of S for each prime ideal P of S.

(iii) $N(P)$ is a left and right symmetric ideal of S for each prime ideal P of S.

(iv) $xy \in N(P)$ implies $ySx \subseteq N(P)$ for $x, y \in S$ and for each prime ideal P of S.

Proof. $(i) \Rightarrow (ii)$: Let S be a 2-primal semiring. Then by Theorem 3.16(v), $N_P = N(P)$ for each prime ideal P of S. We now show that $\overline{N}_P = N_P$ for each prime ideal P of S. Let $a \in \overline{N}_P$. So $a^n \in N_P$ for some positive integer n, which implies that $a^n b \in \mathcal{P}(S)$ for some $b \in S - P$. Since S is a 2-primal semiring, by Theorem 3.16, $\mathcal{P}(S)$ has the IFP and hence $(ab)^n \in \mathcal{P}(S)$. Also by Theorem 3.15, $\mathcal{P}(S)$ is a completely semiprime semiring. Thus $ab \in \mathcal{P}(S)$. So $a \in N_P$. Therefore $N(P) = \overline{N}_P$. We know $N(P) \subseteq N_P \subseteq \overline{N}_P$. Thus $N(P) = N_P = \overline{N}_P$ for each prime ideal P of S. So by Proposition 3.12(ii), $N(P)$ is a completely semiprime ideal of S for each prime ideal P of S.

$(ii) \Rightarrow (iii)$ Let $xyz \in N(P)$, where $x, y, z \in S$. Now $(zxy)^2 = z(xyz)xy \in N(P)$. Since $N(P)$ is completely semiprime, $zxy \in N(P)$. $(xyxz)^2 = xyx(zxy)xz \in N(P) \Rightarrow xyxz \in N(P) \Rightarrow (yxzyx)^2 = yxzy(xyxz)yx \in N(P) \Rightarrow yxzyx \in N(P) \Rightarrow (xzy)^3 = xz(yxzyx)zy \in N(P) \Rightarrow xzy \in N(P)$. Also $(yxz)^2 = y(xzy)xz \in N(P) \Rightarrow yxz \in N(P)$. Therefore $N(P)$ is a left and right symmetric ideal of S.

$(iii) \Rightarrow (iv)$ Let $xy \in N(P)$, where $x, y \in S$. Since $N(P)$ is an ideal of S, for each $s \in S$, $sxy \in N(P)$. As $N(P)$ is right symmetric $syx \in N(P)$. Also since $N(P)$ is left symmetric, $ysx \in N(P)$. Therefore $ySx \subseteq N(P)$.

$(iv) \Rightarrow (i)$ We know $\mathcal{P}(S) \subseteq \mathcal{N}(S)$. Let $x \in \mathcal{N}(S)$, then $x^n = 0$, for some positive integer n. If possible, let $x \notin \mathcal{P}(S)$. Then $x \notin P$ for some prime ideal P of S. Then $x \in S - P$. As P is prime, $S - P$ is an m-system. So there exists s_1 in S such that $xs_1x \in S - P$. Continuing this process there exist $s_2, s_3, \ldots, s_{n-1} \in S$ such that $xs_1xs_2x \ldots xs_{n-1}x \in S - P$. Now by (iv), $x^n = 0 \in N(P)$ implies $xs_1xs_2x \ldots xs_{n-1}x \in N(P)$ i.e. $xs_1xs_2x \ldots xs_{n-1}x \in P$, a contradiction. Thus $x \in \mathcal{P}(S)$. Therefore $\mathcal{P}(S) = \mathcal{N}(S)$. Hence S is a 2-primal semiring. ■

Theorem 3.18. The following statements are equivalent for a semiring:

(i) S is a 2-primal semiring.

(ii) $\overline{O}_P \subseteq P$ for each prime ideal P of S.

(iii) $\mathcal{N}(S) = \bigcap_{P \in Spec(S)} \overline{O}_P = \mathcal{P}(S)$.

Proof. $(i) \Rightarrow (ii)$: Let $a \in \overline{O}_P$. Then there exists a positive integer n such that $a^n \in O_P$. So $a^n b = 0$ i.e. $a^n b \in \mathcal{P}(S)$, for some $b \in S - P$, which implies that $a^n \in N_P$ i.e. $a \in \overline{N}_P$. So $\overline{O}_P \subseteq \overline{N}_P$ for each prime ideal P of S. Also by Theorem 3.16(iv), $\overline{N}_P = N(P) \subseteq P$ for each prime ideal P of S. Thus $\overline{O}_P \subseteq P$ for each prime ideal P of S.

$(ii) \Rightarrow (iii)$: Since $\overline{O}_P \subseteq P$ for each prime ideal P of S, $\bigcap_{P \in Spec(S)} \overline{O}_P \subseteq \bigcap_{P \in Spec(S)} P = \mathcal{P}(S)$. Now by Proposition 3.13, $\mathcal{N}(S) \subseteq \bigcap_{P \in Spec(S)} \overline{O}_P \subseteq \mathcal{P}(S)$. Also $\mathcal{P}(S) \subseteq \mathcal{N}(S)$.

Therefore $\mathcal{N}(S) = \bigcap_{P \in Spec(S)} \overline{O}_P = \mathcal{P}(S)$.

$(iii) \Rightarrow (i)$: Obvious. ■

Theorem 3.19. If $\overline{O}_P = P$ for each prime ideal P of a semiring S, then

(i) S is a 2-primal semiring.

(ii) $\overline{O}_P = N(P)$ for each prime ideal P of S.

(iii) Every prime ideal of S is minimal and completely prime.

Proof. (i) Since $\overline{O}_P = P$, $\overline{O}_P \subseteq P$. Hence by the Theorem 3.18(iii), $\mathcal{N}(S) = \mathcal{P}(S)$ i.e. S is 2-primal.

(ii) Since $N(P) \subseteq P$ and $\overline{O}_P = P$ for each prime ideal P of S, $N(P) \subseteq \overline{O}_P$ for each prime ideal P of S. Now by Theorem 3.16(iv), $N(P) = \overline{N}_P$ for each prime ideal P of S. Also $\overline{O}_P \subseteq \overline{N}_P$ for each prime ideal P of S. Thus $\overline{O}_P \subseteq N(P)$ for each prime ideal P of S. Therefore $\overline{O}_P = N(P)$ for each prime ideal P of S.

(iii) Let P be a prime ideal of S. From (ii) and the given condition $\overline{O}_P = P$, we get $N(P) = P$ for each prime ideal P of S. If Q is a minimal prime ideal of S contained in P, then $N(P) \subseteq N(Q) \subseteq Q \subseteq P = N(P)$. Thus $P = Q$ i.e. P is a minimal prime ideal of S.

Let $xy \in P = N(P)$ and $x \notin P$. Since $xy \in N(P)$, there exists $b \in S - P$ such that $(xy)Sb \subseteq \mathcal{P}(S)$ i.e. $x(ySb) \subseteq \mathcal{P}(S)$. Since $\mathcal{P}(S)$ has the IFP (by Theorem 3.16(ii)), $xS(ySb) \subseteq \mathcal{P}(S) \subseteq P$. As $x \notin P$, $ySb \subseteq P$. Again since $b \notin P$, $y \in P$. So either $x \in P$ or $y \in P$. Hence P is a completely prime ideal of S. ■

Proposition 3.20. If S is a 2-primal semiring and $O_P = P$ for some prime ideal P, then P is a completely prime ideal of S, in particular O_P has the IFP.

Proof. Let $xy \in P = O_P$. If possible, let $x \notin P$. So there exists $b \in S - P$ such that $(xy)b = 0$. Since S is a 2-primal semiring, by Theorem 3.16, $\mathcal{P}(S)$ has the IFP. Therefore $(xSy)Sb \subseteq \mathcal{P}(S) \subseteq P$. Since P is prime and $x \notin P$, $ySb \subseteq P$. Again since $b \notin P$, $y \in P$. Therefore either $x \in P$ or $y \in P$. Hence P is a completely prime ideal of S. ■

Proposition 3.21. Let S be a semiring. If $O(P)$ has the IFP for each minimal prime ideal P of S, then S is a 2-primal semiring.

Proof. Suppose $O(P)$ has the IFP for each minimal prime ideal P of S. To prove S is a 2-primal semiring, it is sufficient to show that $\mathcal{N}(S) \subseteq \mathcal{P}(S)$. Let $a \in \mathcal{N}(S)$. Then $a^n = 0$, for some positive integer n. If possible suppose $a \notin \mathcal{P}(S)$, then there exists a prime ideal P of S such that $a \notin P$. As P is a prime ideal of S, $S - P$

is an m-system of S. So there exists $s_1 \in S$ such that $as_1a \notin P$. Continuing this process, we get $s_2, s_3, \ldots, s_{n-1} \in S$ such that $as_1as_2a \ldots as_{n-1}a \notin P$. Also since $O(P)$ has the IFP, $a^n = 0 \in O(P) \Rightarrow as_1as_2a \ldots as_{n-1}a \in O(P)$. As $O(P) \subseteq P$, $as_1as_2a \ldots as_{n-1}a \in P$, a contradiction. So $a \in \mathcal{P}(S)$. Hence $\mathcal{P}(S) = \mathcal{N}(S)$ i.e. S is a 2-primal semiring. ■

We now prove that if O_P is a prime ideal for each minimal prime ideal P of S, then the converse of the Proposition 3.21 is true.

Proposition 3.22. Assume that O_P be a prime ideal of S for each minimal prime ideal P of S. Then $O(P)$ has the IFP, for each minimal prime ideal P of S if and only if S is a 2-primal semiring.

Proof. Let S be a 2-primal semiring and P be a minimal prime ideal of S such that O_P is a prime ideal of S. So $O_PS \subseteq O_P$ and hence $O_PSb = 0$, for some $b \in S - P$. Thus $O_PSb \subseteq P$. As P is a prime ideal of S and $b \notin P$, $O_P \subseteq P$. Again since O_P is a prime ideal of S and P is a minimal prime ideal of S, $O_P = P$. We now prove that $O(P) = O_P$. Let $x \in O(P)$. Then there exists $y \in S - P$ such that $xSy = 0$. Since S contains the identity element, $xy = 0$. So $x \in O_P$ i.e. $O(P) \subseteq O_P$. Again, let $a \in O_P$. So $aS \subseteq O_P$ and there exists $b \in S - P$ such that $aSb = 0$. Thus $a \in O(P)$. Hence $O_P \subseteq O(P)$. Therefore $O(P) = O_P = P$, for each minimal prime ideal P of S. So by Proposition 3.20, $O(P)$ has the IFP, for each minimal prime ideal P of S. The converse part follows from the Proposition 3.21. ■

Theorem 3.23. Let O_P be a prime ideal of S for each minimal prime ideal of S. Then the following statements are equivalent:

(i) S is a 2-primal semiring.

(ii) O_P has the IFP for each minimal prime ideal P of S.

(iii) O_P is a completely semiprime ideal for each minimal prime ideal P of S.

(iv) O_P is a left and right symmetric ideal for each minimal prime ideal P of S.

(v) $xy \in O_P$ implies $ySx \subseteq O_P$ for $x, y \in S$ and for each minimal prime ideal P of S.

Proof. $(i) \Rightarrow (ii)$ Since O_P is an ideal of S, $O_PS \subseteq O_P$. So $O_PSb = 0$, for some $b \in S - P$. Thus $O_PSb \subseteq P$. As P is a prime ideal of S and $b \notin P$, $O_P \subseteq P$. Again since O_P is a prime ideal of S and P is a minimal prime ideal of S, $O_P = P$. Therefore by Proposition 3.20, O_P has the IFP for each minimal prime ideal P of S.

$(ii) \Rightarrow (iii)$ Let $x^2 \in O_P$. Since by (ii) O_P has the IFP, $xSx \subseteq O_P$. As O_P is a prime ideal of S, $x \in O_P$. Hence O_P is a completely semiprime ideal for each minimal prime ideal P of S.

The proofs of $(iii) \Rightarrow (iv)$ and $(iv) \Rightarrow (v)$ are similar to the proofs of (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) of Theorem 3.15 respectively and so we omit it.

$(v) \Rightarrow (i)$ Let $xy \in \mathcal{P}(S)$. Then $xy \in P$ for each minimal prime ideal P of S. Since O_P is a prime ideal of S for each minimal prime ideal of S, $O_P = P$, for each minimal prime ideal of S. Thus $xy \in O_P$ for each minimal prime ideal P of S. So by (v), $ySx \subseteq O_P = P$ for each minimal prime ideal P of S. Therefore $ySx \subseteq \mathcal{P}(S)$. Hence by (4) $\Rightarrow$ (1) of Theorem 3.15, S is a 2-primal semiring. ■

Theorem 3.24. Let O_P be a prime ideal for each minimal prime ideal P of S. Then the following statements are equivalent:

(i) S is a 2-primal semiring.

(ii) $O(P)$ has the IFP for each minimal prime ideal P of S.

(iii) Every minimal prime ideal of S is a completely prime ideal of S.

Proof. $(i) \Rightarrow (ii)$ Follows from the Proposition 3.22.

$(ii) \Rightarrow (iii)$ Let P be a minimal prime ideal of S. Then O_P is a prime ideal of S and $O(P)$ has the IFP. Then by the proof of the Proposition 3.22, we get $O(P) = O_P = P$. Suppose $xy \in P$. Since $O(P)$ has the IFP, $xSy \subseteq O(P) = P$. Therefore either $x \in P$ or $y \in P$.

$(iii) \Rightarrow (i)$ Since $\mathcal{P}(S)$ is the intersection of all minimal prime ideals of S and by (iii) each minimal prime ideal of S is a completely prime ideal of S and hence $\mathcal{P}(S)$ is the intersection of completely semiprime ideal i.e. $\mathcal{P}(S)$ is a completely semiprime ideal of S. Therefore by Theorem 3.15, S is a 2-primal semiring. ■

Proposition 3.25. Let O_P be a prime ideal of S for every minimal prime ideal P of S. Then S is a 2-primal semiring if and only if $P = \overline{O(P)} = \overline{O}_P$ for each minimal prime ideal P of S.

Proof. Suppose S is a 2-primal semiring. Since O_P is a prime ideal for each minimal prime ideal P of S, by the proof of the Proposition 3.22, $P = O(P) = O_P$. Therefore by Proposition 3.20, $O_P = O(P)$ is a completely prime and hence a completely semiprime ideal of S. Hence by Proposition 3.12(ii), $O_P = \overline{O}_P$ i.e. $P = \overline{O(P)} = \overline{O}_P$.

Conversely, suppose $P = \overline{O(P)} = \overline{O}_P$ for each minimal prime ideal P of S. By Proposition 3.13, $\mathcal{N}(S) \subseteq \bigcap_{P \in Spec(S)} \overline{O}_P \subseteq \bigcap_{Q \in mSpec(S)} \overline{O}_Q = \bigcap_{Q \in mSpec(S)} Q = \mathcal{P}(S)$. Also $\mathcal{P}(S) \subseteq \mathcal{N}(S)$. So $\mathcal{P}(S) = \mathcal{N}(S)$. Hence S is a 2-primal semiring. ■

Proposition 3.26. If O_P has the IFP for each minimal prime ideal P of S, then $O_P \subseteq P$ for each minimal prime ideal P of S if and only if S is a 2-primal semiring.

Proof. Suppose $O_P \subseteq P$ for each minimal prime ideal P of S. Let $a \in \mathcal{N}(S)$. Then $a^n = 0$, for some positive integer n such that $a^n = 0$. If possible let $a \notin \mathcal{P}(S)$. So there exists a prime ideal P of S such that $a \notin P$. Since P is a prime ideal of S, $S - P$ is a m-system and therefore there exist $s_1, s_2, \dots, s_{n-1} \in S$ such that $as_1as_2a \dots s_{n-1}a \notin P$. Again since O_P has the IFP and $a^n = 0 \in O_P$, $as_1as_2a \dots s_{n-1}a \in O_P$. Since $O_P \subseteq P$, $as_1as_2a \dots s_{n-1}a \in P$, a contradiction. So $a \in \mathcal{P}(S)$. Hence $\mathcal{N}(S) \subseteq \mathcal{P}(S)$. Also $\mathcal{P}(S) \subseteq \mathcal{N}(S)$. Thus $\mathcal{P}(S) \subseteq \mathcal{N}(S)$ i.e. S is a 2-primal semiring.

Conversely, suppose S is a 2-primal semiring and P is a minimal ideal of S. Let $a \in O_P$. Then there exists $b \in S - P$ such that $ab = 0$. So $ab \in \mathcal{P}(S)$. Since S is a 2-primal semiring, by Theorem 3.16, $\mathcal{P}(S)$ has the IFP. So $aSb \subseteq \mathcal{P}(S)$ which implies that $aSb \subseteq P$. Since P is a prime ideal of S and $b \notin P$, $a \in P$. Hence $O_P \subseteq P$. ■

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