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Some Results on n-Normal and Hyponormal Operators on the Tensor Product of Hilbert Spaces

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Abstract

In this paper, we have studied n-Normal and hyponormal operators in the tensor product of Hilbert spaces. Let X, Y be Hilbert spaces and T_1 be an operator on X. Let K be a closed subspace of Y and p be a projection of Y onto K. Using T_1 and the projection p, we construct an operator T on the tensor product $X \otimes_\gamma Y$. Different conditions under which the operator becomes n-Normal, hyponormal, M-hyponormal and quasi M-hyponormal are derived here. Several interesting results regarding the spectral radius and the null space of T are also discussed here.

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1 Introduction

The classes of n-Normal and Hyponormal operators have received important consideration in the study of operator theory in Hilbert spaces. Several interesting aspects regarding these operators are investigated by many outstanding research workers (refer to[2], [5], [8], [9]). In this paper, we study such kind of operators in the projective tensor product of two Hilbert spaces and derive some interesting results.
Before going to prove our main results, we present some basic definitions and preliminary lemmas.

1.1 Algebraic Tensor Product

Let X, Y be normed spaces over $\mathbb{K}$ with dual spaces X^*, Y^*. Given $x \in X, y \in Y$, let $x \otimes y$ be the element of $BL(X^*, Y^*; \mathbb{K})$, the set of all bounded bilinear functional from $X^* \times Y^*$ to $\mathbb{K}$, and is defined by $(x \otimes y)(f,g) = f(x)g(y)\ (f \in X^*, g \in Y^*)$
The algebraic tensor product of X and Y, $X \otimes Y$ is defined to be linear span of $\{x \otimes y : x \in X, y \in Y\}\ in\ BL(X^*, Y^*; \mathbb{K})$. (refer to [4])

1.2 Projective Tensor Product

Given normed spaces X and Y, the projective tensor norm γ on $X \otimes Y$ is defined by

$$\gamma(u) = \inf\left\{\sum_i \| x_i \| \| y_i \| : u = \sum_i x_i \otimes y_i\right\}$$

wherethe infimum is taken over all finite representations of u.
The completion of $X \otimes Y$ with respect to γ is called the projective tensor product of X and Y and is denoted by $X \otimes_\gamma Y$ (refer to [4]).
If X and Y are Hilbert spaces, an inner product on $X \otimes_\gamma Y$ is defined as $\langle a \otimes b, c \otimes d\rangle = \langle a, c\rangle\langle b, d\rangle, a, c \in X$ and $b, d \in Y$.
Then it can be shown that $X \otimes_\gamma Y$ is a Hilbert space.

1. 3 n-Normal Operators

An operator $T \in B(H)$ (the set of all bounded linear operators on a Hilbert space H) is called an n-normal operator if $T^n T^* = T^* T^n$.
An operator $T \in B(H)$ is said to be quasi n-normal operator if $T(T^n T^*) = (T^* T^n)T$.

1. 4Hyponormal Operators

An operator T on a Hilbert Space H is said to be hyponormal if $T^* T \geq T T^*$ which is equivalent to $\| T^*(x) \| \leq \| T(x) \|, \forall x \in H$. An operator T is quasi hyponormal if $(T^*)^2 T^2 \geq (T^* T)^2$ holds, which is equivalent to $\| T^* T(x) \| \leq \| T^2(x) \|, \forall x \in H$.
An operator T is said to be M- hyponormal if there exists a positive real number M such that $M^2 (T - \lambda)^*(T - \lambda) \geq (T - \lambda)(T - \lambda)^*$ for all $\lambda \in \mathbb{C}$.
In general, hyponormal⇒M-hyponormal⇒quasi M-hyponormal. [Refer to [15]]

Lemma 1. 1. [Fuglede Theorem][10]: If T is an operator then $T^n T = T T^n$ for any n.

Lemma 1. 2. [10] Let $T \in B(H)$. Then T is n-normal if and only if T^n is normal.

Lemma 1. 3. [10] Let $T \in B(H)$. If T is both 2 and 3-normal then T is n-normal.

Lemma 1. 4. [2] Let $T \in B(H)$. Then T is hyponormal operator if and only if $T^*T + 2\lambda TT^* + \lambda^2 TT^* \geq 0 \; \forall \lambda \in \mathbb{R}$.

Lemma 1. 5. [2] T is quasi M-hyponormal if and only if there exists a positive real number M such that $M \| (T - \lambda)Tx \| \geq \| (T - \lambda)^*Tx \| \; \forall x \in H \text{ and } \forall \lambda \in \mathbb{C}$.

Lemma 1. 6. [2] T is M-hyponormal if and only if there exists a positive real number M such that $M \| (T - \lambda)x \| \geq \| (T - \lambda)^*x \|; \; \forall x \in H \text{ and } \forall \lambda \in \mathbb{C}$.

Lemma 1. 7. [14] If T is M-hyponormal on H and $K \subset H$ is a closed T-invariant subspace, then $T|_K$ is M-hyponormal.

2 Main Results

First, we construct an operator T on the projective tensor product of Hilbert spaces X and Y. Let T_1 be an operator on X. Let K be a closed subspace of the Hilbert Space Y and p be a projection of Y onto K.

We define $T : X \otimes_\gamma Y \rightarrow X \otimes_\gamma Y$

by $T(\Sigma_i x_i \otimes y_i) = \Sigma_i T_1(x_i) \otimes p(y_i); \; \Sigma_i x_i \otimes y_i \in X \otimes_\gamma Y$

Theorem 2. 1. *The operator T defined above has the following properties*

a. If T_1 is 2 and 3 normal then T is n-normal.

b. If T_1 is quasi n-normal then T is quasi n-normal.

Proof. (a) We have, $T\left(\sum_i x_i \otimes y_i\right) = \sum_i T_1(x_i) \otimes p(y_i)$

Now,

$T^2(\Sigma_i x_i \otimes y_i) = T.T(\Sigma_i x_i \otimes y_i)$

$= T(\Sigma_i T_1(x_i) \otimes p(y_i))$

$= \Sigma_i T_1^2(x_i) \otimes p(y_i) \; \ldots\ldots\ldots (2.1)$

Since T_1 is 2-normal so, $T_1^2 T_1^* = T_1^* T_1^2$.

But, $T^*(\Sigma_i x_i \otimes y_i) = \Sigma_i T_1^*(x_i) \otimes p(y_i)$

Now, using (2. 1),

$T^*T^2(\Sigma_i x_i \otimes y_i) = T^*\left(\Sigma_i T_1^2(x_i) \otimes p(y_i)\right)$

$= \Sigma_i T_1^* T_1^2(x_i) \otimes p(y_i)$

$= \Sigma_i T_1^2 T_1^*(x_i) \otimes p(y_i)$

$= T^2 T^*(\Sigma_i x_i \otimes y_i) \; \ldots\ldots\ldots (2.2)$

Therefore, T is 2-normal.

Since T_1 is 3-normal we have, $T_1^3 T_1^* = T_1^* T_1^3$

Now, as in (2. 2) we can show that $T^*T^3(\Sigma_i x_i \otimes y_i) = T^3 T^* (\Sigma_i x_i \otimes y_i)$

Therefore, T is 3-normal.

As T is both 2-normal and 3-normal so by Lemma[1. 3] T is n-normal.

Proof. (b) $T(T^nT^*)(\Sigma_i x_i \otimes y_i) = TT^n\Sigma_i T_1^*(x_i) \otimes p(y_i)$
$= T\Sigma_i T_1^n T_1^*(x_i) \otimes p(y_i)$
$= \Sigma_i T_1(T_1^n T_1^*)(x_i) \otimes p(y_i)$
$= \Sigma_i (T_1^* T_1^n)T_1(x_i) \otimes p(y_i)$, [$\because T_1$ is quasi n-normal]
$= T^*T^n\big(T(\Sigma_i x_i \otimes y_i)\big)$
Thus $T(T^nT^*) = (T^*T^n)T$
Therefore, T is quasi n-normal. □
Now, we discuss hyponormality conditions for the operator T.

Theorem 2. 2. *T ishyponormal if and only if T_1 is hyponormal.*

Proof. If T_1 is hyponormal, then $\| T_1^*(x) \| \leq \| T_1(x) \| \ \forall x \in X$
Now, $\| T^*(\sum_i x_i \otimes y_i) \| = \| \sum_i T_1^* x_i \otimes p(yi) \|$
$\leq \sum_i \| T_1^* x_i \| \| p(y_i) \|$
$\leq \sum_i \| T_1 x_i \| \| p(y_i) \|$ [$\because T_1$ is hyponormal]
Thus, $\| T^*(\sum_i x_i \otimes y_i) \| \leq \| T(\sum_i x_i \otimes y_i) \|$ [Taking projective norm]
So, T is hyponormal.
Conversely, Let T be hyponormal.
We fix an element m in K.
Then $m = p(m)$; $m \neq 0$
Let $x \in X$ be arbitrary.
Now, $\| T_1^*(x) \| . \| m \| = \| T_1^*(x) \| \| p^*(m) \|$
$= \| T_1^*(x) \otimes p^*(m) \|$
$= \| T^*(x \otimes m) \|$
$\leq \| T(x \otimes m) \|$ [$\because T$ ishyponormal]
$= \| T_1(x) \otimes p(m) \|$
$= \| T_1(x) \| \| p(m) \|$
$= \| T_1(x) \| \| m \|$
$\Rightarrow \| T_1^*(x) \| \leq \| T_1(x) \|$, showing that T_1 is hyponormal. □
In view of the above theorem we get the following corollary:

***Corollary* 2. 0. 1.** For all $\lambda \in \mathbb{R}$
$T^*T + 2\lambda TT^* + \lambda^2 TT^* \geq 0 \Leftrightarrow T_1^*T_1 + 2\lambda T_1^*T_1 + \lambda 2T_1^*T_1 \geq 0$
Similar result holds in case of quasi M-hyponormality condition also.

Theorem 2. 3. *T_1 is quasi M-hyponormal if and only if T is quasi M-hyponormal.*

Proof. Let T be quasi M-hyponormal. So, by Lemma 1. 5,
$M \| (T-\lambda)Tu \| \geq \| (T-\lambda)^* Tu \|, \forall u \in X \otimes_\gamma Y$ and $\forall \lambda \in \mathbb{C}$--------(2. 3)
Now if $m \in K$, then $m = p(m)$, $(m \neq 0)$
For $x \in X$,
$M \| (T_1 - \lambda)T_1(x) \| \| m \| = M \| T_1(T_1 x) - \lambda T_1(x) k \| \| p(m) \|$
$= M \| (T_1(T_1 x) - \lambda T_1(x)) \otimes p(m) \|$
$= M \| T(T_1 x \otimes m) - \lambda T(x \otimes m) \|$
$= M \| (T-\lambda)T(x \otimes m) \|$
$\geq \| (T-\lambda)^* T(x \otimes m) \|$ $using\,(2.3)$
$= \| (T^* - \bar{\lambda})T(x \otimes m) \|$
$= \| T^*((T_1 x) \otimes p(m)) - \bar{\lambda} T_1 x \otimes p(m) \|$
$= \| (T_1^* - \bar{\lambda})T_1 x \otimes p(m) \|$
$= \| (T_1 - \lambda)^* T_1 x \| \| p(m) \|$
$= \| (T_1 - \lambda)^* T_1 x \| \| m \|$
$\Rightarrow M \| (T_1 - \lambda)T_1 \| \geq \| (T_1 - \lambda)^* T_1 x \|$ $(\because m \neq 0)$,
showing that T_1 is quasi M-hyponormal.
Conversely, let T_1 be quasi M-hyponormal.
For $\sum_i x_i \otimes y_i \in X \otimes_\gamma Y$ we have,
$\| (T-\lambda)^* T(\Sigma_i x_i \otimes y_i) \| = \| (T^* - \lambda)\left(\Sigma_i T_1(x_i) \otimes p_1(y_i)\right) \|$
$= \| \Sigma_i T_1^* T_1(x_i) \otimes p_1(y_i) - \lambda \Sigma_i T_1(x_i) \otimes p_1(yi) \|$
$= \| \Sigma_i (T_1^* - \lambda) T_1(x_i) \otimes p_1(y_i) \|$
$\leq \Sigma_i \| (T_1 * - \lambda) T_1(x_i) \| . \| p_1(y_i) \|$
$= \Sigma_i \| (T_1 - \lambda)^* T_1 x_i \| . \| p_1 y_i \|$
$\leq \Sigma_i M \| (T_1 - \lambda) T_1 x_i \| \| p_1 y_i \|$
[$\because T_1$ is quasi M-hyponormal]
So, taking projective tensor norm,
$\| (T-\lambda)^* T\,(\Sigma_i x_i \otimes y_i) \| \leq M \| \Sigma_i (T_1 - \lambda) T_1 x_i \otimes p_1(y_i) \|$
$= M \| \Sigma_i (T_1)^2(x_i) \otimes p_1(y_i)) - \lambda T\,(\Sigma_i x_i \otimes y_i) \|$
$= M \| T^2(\Sigma_i x_i \otimes y_i) - \lambda T\,(\Sigma_i x_i \otimes y_i) \|$
$= M \| (T-\lambda)T\,(\Sigma_i x_i \otimes y_i) \|$
Thus $M \| (T-\lambda)T\,(\Sigma_i x_i \otimes y_i) \| \geq \| (T_1 - \lambda)^* T\,(\Sigma_i x_i \otimes y_i) \|$
Therefore, T is quasi M-hyponormal. □

Theorem 2. 4. *T_1 is M-hyponormal if and only if T is M-hyponormal.*
The proof is similar as above.
Now we consider a closed subset of the Hilbert space X and derive the following

result.

Theorem 2. 5. *If* T_1 *is a quasi* M*-hyponormal operator on a closed subset* J *of* X*, then* T *is* M*-hyponormal on* $J \otimes K$.

Proof. Since T_1 is quasi M-hyponormal on J, so, by lemma 1. 5

$M \| (T_1 - \lambda)T_1 x \| \geq \| (T_1 - \lambda)^* T_1 x \| \ \forall x \in J$ and $\lambda \in C$

For $\Sigma_i j_i \otimes k_i \in J \otimes K$

$\| (T - \lambda)^*(\Sigma_i j_i \otimes k_i) \| = \| \Sigma_i (T_1 - \lambda)^* j_i \otimes p_1 k_i \|$

$\leq \Sigma_i \| (T_1 - \lambda)^* j_i \| \| p_1 k_i \|$

$\leq \Sigma_i M \| (T_1 - \lambda) T_1 j_i \| \| p_1 k_i \|$

$= M \Sigma_i \| (T_1 - \lambda) T_1 j_i \| \| p_1 k_i \|$.

Taking the projective tensor norm,

$\| (T - \lambda)^*(\Sigma_i j_i \otimes k_i) \| \leq M \Sigma_i \| (T_1 - \lambda) T_1 j_i \otimes p_1 k_i \|$

$= M \| (T - \lambda)(\Sigma_i j_i \otimes k_i) \|$

Thus, $M \| (T - \lambda)\Sigma_i j_i \otimes k_i) \| \geq \| (T - \lambda))^*(\Sigma_i j_i \otimes k_i) \|$

so, using the Lemma 1. 6, we have, T is M-hyponormal on $J \otimes K$.

Theorem 2. 6. *If* T *is* M*-hyponormal and* Q *is a* T*-invariant closed subspace of* $X \otimes_\gamma Y$*, then corresponding to* Q *there exists a closed subspace* $\hat{Q}$ *of* X *such that,* $T_1|_{\hat{Q}}$ *is* M*-hyponormal.*

Proof. For $g \in Y^*$(The dual space of Y), we define, $F : X \otimes \gamma Y \to X$ such that $F(\Sigma_i x_i \otimes y_i) = \Sigma_i g(y_i) x_i$, $\Sigma_i x_i \otimes y_i \in X \otimes \gamma Y$

Then F is a bounded linear operator on $X \otimes_\gamma Y$ *with* $\| F \| \leq \| g \|$.

Let $R = Q \cap (X \otimes K)$. Then R is a closed subspace of $X \otimes_\gamma Y$.

Now, $T(Q) \subset Q \Rightarrow F(T(Q)) \subset F(Q)$.

So, for $\Sigma_i x_i \otimes y_i \in Q$, we have,

$F(T(\Sigma_i x_i \otimes y_i)) \in F(Q)$

i. e., $F(\Sigma_i T_1 x_i \otimes p_1(y_i)) \in F(Q) \Rightarrow \Sigma_i g(p_1 y_i) T_1 x_i \in F(Q)$.

Let, $\hat{Q} = F(R)$

$= \{F(\Sigma_i q_i \otimes r_i) : \Sigma_i q_i \otimes r_i \in R\}$

$= \{\Sigma g(r_i) q_i : \Sigma_i q_i \otimes r_i \in R\}$.

For, $\Sigma_i a_i \otimes b_i \in R = Q \cap (X \otimes K)$, we have, $b_i \in K \ \forall i$. So, $p_1 b_i = b_i \forall i$

Also,

$T_1(F(\Sigma_i a_i \otimes b_i)) = T_1(\Sigma_i g(b_i) a_i)$

$= \Sigma_i g(b_i) T_1 a_i$

$= \Sigma_i g(p_1 b_i) T_1 a_i$

$= F(\Sigma_i T_1 a_i \otimes p_1 b_i)$
$= F\big(T(\Sigma_i a_i \otimes b_i)\big).$
Since $\Sigma_i a_i \otimes b_i \in Q$ therefore, $T(\Sigma_i a_i \otimes b_i) \in Q$ (as $T(Q) \subset Q$)
Again,
$T(\Sigma_i a_i \otimes b_i) = \Sigma_i T_1 a_i \otimes p_1 b_i = \Sigma_i T_1 a_i \otimes b_i \in X \otimes K.$
Thus,
$T(\Sigma_i a_i \otimes b_i) \in Q \cap (X \otimes K) = R$
Therefore, $T_1\big(F(\Sigma_i a_i \otimes b_i)\big) \in F(R) = \tilde{Q}.$
By Theorem 2. 4 $T_1(\tilde{Q}) \subset \tilde{Q}$
So, $\tilde{Q}$ is a T_1 invariant closed subspace of X.
Now, T is M-hyponormal $\Rightarrow T_1$ is M-hyponormal.
So, by Lemma 1. 7 $T_1|_{\tilde{Q}}$ is M-hyponormal. □
Next we want to discuss spectral properties of the operator T.
The spectral radius, denoted by $r(T)$ is defined as $r(T) = \sup\{|z| : z \in \sigma(T)\}$ where $\sigma(T)$ denotes the spectrum of the operator T.
The numerical range, denoted by $W(T)$ is defined as $W(T) = closure\{z : z = \langle Tx, x\rangle, \| x \| = 1\}$

Lemma 2. 1. ([13]) If T is hyponormal and $(T - zI)^{-1}$ exists(as a bounded operator) $(z \in \mathbb{C})$ then $(T - zI)^{-1}$ is hyponormal.

Lemma 2. 2. ([13]) Let T be hyponormal with z contained in the resolvent set of T, *i. e.*, $z \in \Omega(T)$. Then $\| (T - zI)^{-1} \| \leq \frac{1}{d(z,\sigma(T))}$

Lemma 2. 3. ([13]) For any operator T, $\| (T - zI)x \| \geq d\big(z, W(T)\big)$, $\| x \| = 1$.

Theorem 2. 7. *If T is M-hyponormal, then for any $\lambda \in \mathbb{C}$, spectral radius of $(T - \lambda)$,*
$r(T - \lambda) \leq M^{\frac{1}{2}} \| (T - \lambda) \|$

***Proof*.** For $n \in \mathbb{N}$ and $u \in X \otimes \gamma Y$
$\| (T - \lambda)^n u \|^2 = \langle (T - \lambda)^n u, (T - \lambda)^n u\rangle$
$= \langle (T - \lambda)^*(T - \lambda)^n u, (T - \lambda)^{n-1} u\rangle$
$\leq \| (T - \lambda)^*(T - \lambda)^n u \| \| (T - \lambda)^{n-1} u \|$
$\leq M \| (T - \lambda)^{n+1} u \| \| (T - \lambda)^{n-1} u \|$ [$\because T$ is M-hyponormal]
$\leq M \| (T - \lambda) \|^{n+1} \| u \| \| (T - \lambda) \|^{n-1} \| u \|$
\= $M \| (T - \lambda) \|^{2n} \| u \|^2$ (2. 4)

For n=1 from 2. 4 we have

$\| (T - \lambda)u \|^2 \leq M \| (T - \lambda) \|^2 \| u \|^2$

$\Rightarrow \| (T - \lambda)u \| \leq M^{\frac{1}{2}} \| (T - \lambda) \| \| u \|$

So, $\| (T - \lambda)^n u \|^2 \leq M[M \| (T - \lambda)^2 \|]^n \| u \|^2$

$= M^{n+1} \| (T - \lambda) \|^{2n} \| u \|^2$

$\Rightarrow \| (T - \lambda)^n u \| \leq M^{\frac{n+1}{2}} \| (T - \lambda) \|^n \| u \|$

$\Rightarrow \| (T - \lambda)^n \| \leq M^{\frac{n+1}{2}} \| (T - \lambda) \|^n$

Now, $r(T - \lambda) = \lim_{n\to\infty} \| (T - \lambda)^n \|^{\frac{1}{n}}$

$\leq \lim_{n\to\infty} M^{\frac{n+1}{2n}} \| (T - \lambda) \|$

$= M^{\frac{1}{2}} \| (T - \lambda) \|$ □

Theorem 2. 8. *Let* T *be M-hyponormal with* $\mu \notin \sigma(T)$. *Then* $\| (T - \mu I)^{-1} \| \geq \frac{1}{\sqrt{M d(\mu, \sigma(T))}}$; *provided* $(T - \mu I)^{-1}$*exists.*

***Proof*.** Using Theorem 2. 6, for $\| (T - \mu I)^{-1} \|$, we have,

$\| (T - \mu I)^{-1} x \| \geq \frac{1}{\sqrt{M}} r((T - \mu I)^{-1})$

$= \frac{1}{\sqrt{M}} max\{|u| : u \in \sigma((T - \mu I) - 1)\}$

$= 1/[\sqrt{M} \, min\{|u| : u \in \sigma(T - \mu I)\}]$

$= 1/[\sqrt{M} \, min\{|u - \mu| : u \in \sigma(T)\}]$

$= \frac{1}{\sqrt{M} d(\mu, \sigma(T))}$ □

Theorem 2. 9. $N(T_1 - \lambda) \otimes K \subseteq N(T - \lambda) \; \forall \lambda \in C$, *where* $N(T)$*denotes the null space of* T.

***Proof*.** Let $x \otimes m \in N(T_1 - \lambda) \otimes K$.

Then $x \in N(T_1 - \lambda)$ and $m \in K$.

$\therefore T_1 x = \lambda x$

Now, $T(x \otimes m) = T_1(x) \otimes p(m) = \lambda x \otimes m = \lambda(x \otimes m)$

$\therefore x \otimes m \in N(T - \lambda)$

$\therefore N(T_1 - \lambda) \otimes K \subseteq N(T - \lambda)$. □

For *M*-hyponormal operator $T, N(T - \lambda) \subseteq N(T - \lambda)^* \forall \lambda \in \mathbb{C}$.

So, we have, $N(T_1 - \lambda) \otimes M \subseteq N(T - \lambda) \subseteq N(T - \lambda)^*$

Concluding Remark:

There are some other aspects to be investigated regarding thin spectra, integral decomposition, growth condition etc., (refer to [1], [5]) for all the above types of operators on Hilbert Spaces. In [2], [7] there are some interesting results regarding paranormal operators in Hilbert Spaces. An operator T on a Hilbert Space H is said to be *paranormal* if $\| T(x) \|^2 \leq \| T^2(x) \|$, for every unit vector x in H.

Now the following problem can be raised: Can we derive analogous results in case of paranormal operators on $X \otimes_\gamma Y$?

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