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Some Properties of the Chaotic Shift Map On the Generalised m-Symbol Space and Topological Conjugacy

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Abstract

The aim of this paper is to establish some results in connection with the chaotic behaviours of the forward shift map σ^+ on the generalised one-sided symbol space Σ_m^+, $m(\geq 2) \in N$. We prove that σ^+ is *Devaney chaotic*, *Auslander –Yorke's chaotic* and *generically* δ*-chaotic.* We also prove that σ^+ is *exact Devaney chaotic* and as a consequence *mixing Devaney Chaotic* and *weak mixing Devaney Chaotic*. It is further established that the shift map σ^+ on Σ_m^+ is topologically conjugate to the map $f_m(x) = mx(\text{mod}1)$ on the space R/Z or equivalently conjugate to the map $f_m(z) = z^m$ on the circle S^1.

Keywords: Shift Map, Exact Map, Topological Transitivity, Topological Mixing, Topological Conjugacy.

2010 Subject Classifications: 37B10, 37D45, 37C15

1.Introduction:

Discretization of spaces gives birth to symbolic dynamics which is a very powerful tool to analyse general dynamical systems in more effective ways. In fact, symbolic dynamics is a topological dynamical system (X, f) where X is a compact metric space and $f : X \to X$ is a continuous transformation. That is, $f \in C(X) = C(X, X)$, the space of all continuous maps from X into X equipped with the sup norm. It is to be noted that $C(X)$ is a complete and separable metric space.

The chaotic behaviour of topological dynamical systems attracted the attention of the researchers since the introduction of the most popular but curious word 'chaos' in

1975 by *T.Y. Li* and *James A. Yorke* in their much stated paper 'Period three implies chaos'[11]. The basic ingredients of *Li-Yorke chaos* (as it is known today) are uncountable scrambled sets and so called *Li-Yorke pairs*. In last few decades a quite good quantum of knowledge has been added in the field of dynamical systems which includes some innovative ideas and stronger ingredients of analysis such as *transitivity, mixing, sensitivity dependence on initial conditions, exactness* etc. In 1980, *J. Auslander* and *J.A. Yorke* [1] defined chaos by associating the concept of transitivity and sensitivity which is known as *Auslander-Yorke chaos*. Thereafter, in 1989, *R.L. Devaney* [7] posed the example of defining chaos in an another way to study deterministic processes. The chaos defined by *Devaney* is known as *Devaney chaos* today. A map f on a closed set X is called *chaotic in Devaney's sense* or *Devaney chaotic* (*DevC* in short) if

(i) f is transitive on X,

(ii) the set $P(f)$ of all periodic points of f is dense in X, and

(iii) f exhibits sensitive dependence on initial conditions[2, 7].

Devaney chaos is important in analytical as well as application point of view. In the year 1992, *L. Snoha* [21] introduced the concept of *dense chaos* as well as *dense δ-chaos*. The notion of *distributional chaos* was then introduced in 1994 by *Schweizer* and *Smital* [19] and thereafter *L. Wang* gave the concept of *distributional chaos in a sequence* in 2007[22]. Today, *Devaney chaos* has been studied in more extensive ways i.e. *Devaney chaos* with much stronger conditions such as *EDevC*, *MDevC* and *WMDevC.* A *Devaney chaotic* map $f: X \to X$ is said to exhibit *exact Devaney chaos* (*EDevC*) [10] if it is *exact*, i.e., if for every non-empty open set U of X, there exists some $m \in N$ such that $f^m(U) = X$. On the other hand, a *Devaney chaotic* map $f: X \to X$ is said to exhibit *mixing Devaney chaos* (*MDevC*), if it is mixing, i.e., for any pair of non-empty open sets U and V of X, $\exists n_0 \in N$ such that $f^n(U) \bigcap V \neq \phi$, $\forall n \geq n_0$. Further, the *Devaney chaotic* map $f: X \to X$ is said to exhibit *weak mixing Devaney chaos* (*WMDevC*), if it is weak mixing i.e. $f \times f$ is transitive on $X \times X$ i.e. for every non-empty open sets U, V of $X \times X$, $\exists n \in N$ such that $f^n(U) \bigcap V \neq \phi$. In this paper, we have given a formal description of the generalised one-sided m-symbol sequence space Σ_m^+ = {0, 1, 2, …m-1}$^{\mathrm{N}}$ = $\{(x_i)_{i=1}^{\infty} : x_i \in \{0,1,2,.....,m-1\}, m \in N\}$, and discussed the Devaney chaoticity of the forward shift map $\sigma^+ : \Sigma_m^+ \to \Sigma_m^+$. For $m = 2$, we have the symbolic dynamical system Σ_2^+ which is a well known example of a chaotic dynamical system. Many works have been done on this system and hence many references of the binary sequence space Σ_2^+ and the forward shift map σ^+ can be found in various papers and books such as [3, 7, 8, 9, 12, 16, 17, 20]. It is easy to understand that $\Sigma_2^+ \subseteq \Sigma_m^+$ and hence we can certainly expect that some of the results in

(Σ_2^+, σ^+) may be extended to (Σ_m^+, σ^+). In fact, we have extended some of the concepts available in the symbolic dynamical system (Σ_2^+, σ^+) to (Σ_m^+, σ^+). Here, we have established that the forward shift map on the generalised *m*-symbol space Σ_m^+ is *DevC* and as a consequence it is chaotic in *Auslander-Yorke's* sense. Further it is proved that σ^+ is *EDevC (Exact Devaney chaotic)* and consequently *MDevC (Mixing Devaney chaotic)* and *WMDevC (Weakly mixing Devaney chaotic)map*. Further, we have proved that $\sigma^+ : \Sigma_m^+ \to \Sigma_m^+$ is topologically conjugate to the map $f_m(x) = mx(\text{mod}1)$ on the space R/Z.

Note: In the whole paper we have used the "shift map" in place of "forward shift map" and the symbols σ and Σ in lieu of σ^+ and Σ_m^+ respectively for the sake of simplicity. Also, by N we have denoted the set of natural numbers.

2. Basic Concepts and Results:

Definition 2.1: ***Auslander-Yorke chaotic maps*** [13]: A continuous map f on a metric space X is said to be ***chaotic*** according to *Auslander* and *Yorke* if f is transitive and f has sensitive dependence on initial conditions. So, it immediately follows that a *DevC* map is always *chaotic in Auslander-Yorke's sense*.

Definition 2.2: Li-YorkePair [4]: A pair $(x, y) \in X^2$ is called a Li-Yorke pair with modulus $\delta > 0$ if $\lim_{n\to\infty} Sup\, d(f^n(x), f^n(y)) \geq \delta$ and $\lim_{n\to\infty} Inf\, d(f^n(x), f^n(y)) = 0$, where f is a continuous transformation on the compact metric space (X, d). The set of all *Li-Yorke pairs* of modulus $\delta > 0$ is denoted by $LY(f, \delta)$.

Definition 2.3: Weakly and **modified weakly chaotic dependence on initial conditions[3]**: A dynamical system (X, f) is called weakly(resp. modified weakly) chaotic dependence on initial conditions if for any $x \in X$ and every neighbourhood $N(X)$ of x, there are $y, z \in N(X)$ [in modified weakly case $y \neq x, z \neq x$] such that $(y, z) \in X^2$ is *Li-Yorke*.

Definition 2.4: Generically δ-Chaotic maps[3]: The continuous map $f : X \to X$ on a compact metric space X is called *generically δ-chaotic* if $LY(f, \delta)$ is residual in X^2.

Proposition.2.5[7]: A topological dynamical system(TDS) $f : X \to X$ is topologically transitive if for every pair of non-empty open sets U and V of X, there exists $n \in N$ such that $f^n(U) \cap V \neq \phi$.

Proposition 2.6[6]: Let X be a compact metric space and $T : X \to X$ be a continuous topologically mixing map. Then T is also topologically weak mixing.

Proposition2.7[18]: Let $T: X \to X$ be a continuous map on a compact metric space X. If T is topologically weak mixing, then it is generically δ-chaotic on X with $\delta = diam(X)$.

2.8: The topological dynamical system (Σ_m, σ):

It is well known that $\Sigma_2 = \{0,1\}^N = \{a = (a_i)_{i=1}^{\infty} : a_i \in \{0,1\}\}$ is a compact metric space for the metric d defined by $d(x,y) = \sum_{k\geq 1} \frac{|x_k - y_k|}{2^k}$ where $x = (x_1, x_2, x_3, x_4, ...) \in \Sigma_2$, and $y = (y_1, y_2, y_3, y_4, ...) \in \Sigma_2$. Also the shift map σ on Σ_2 defined by $\sigma(a_1, a_2, a_3, ...) = (a_2, a_3, a_4, ...)$ is a continuous map[3]. Thus (Σ_2, σ) is a topological dynamical system(TDS) which is chaotic in nature.

Similarly, the space $\Sigma_m = \{0,1,2,....., m-1\}^N = \{x = (x_i)_{i=1}^{\infty} : x_i \in \{0,1,2,....., m-1\}\}$ where $m(\geq 2) \in N$, the generalised m-symbol sequence space, is a compact metric space under the distance function $d: \Sigma_m \times \Sigma_m \to R$ defined by

$$d(x,y) = \sum_{k\geq 1} \frac{|x_k - y_k|}{m^k} \text{ for } x = (x_1, x_2, x_3, ...),\; y = (y_1, y_2, y_3, ...) \in \Sigma_m$$

It is to be noted that d is always a metric if we replace m in the definition of d by any number $\rho > m$. Again, it can easily be seen that the shift transformation $\sigma: \Sigma_m \to \Sigma_m$ defined by $\sigma(x_1, x_2, x_3, x_4, ...) = (x_2, x_3, x_4, ...)$ is continuous. Therefore, (Σ_m, σ) is clearly a topological dynamical system. One important fact is that for any $m, n(< m) \in N$, we always have that $\Sigma_n \subseteq \Sigma_m$.

2.9 : The map $f_m(x) = mx$ mod1 and the unit circle S^1:

It is a well known fact that for any $m(\geq 2) \in N$, the map $f_m : [0,1] \to [0,1]$ s.t. $f_m(x) = mx$ (mod1) is discontinuous at the points $\frac{1}{m}, \frac{2}{m}, \frac{3}{m}, ..., \frac{m-1}{m} \in I$. This map is well defined on $R/Z \cong I/\sim$ where R/Z is the space of equivalence classes $x + Z$ of real numbers x up to integers such that two real numbers x, y belong to the same equivalence class if an only if $\exists$ an integer $k \in Z$ such that $x = y + k$ and $I/\sim$ denotes the unit interval with the endpoints identified such that the symbol ~ implies that 0~1 are glued together to get a circle. R/Z is a metric space under the metric $d(x,y) = \min_{m\in Z} |x - y + m|$.

Also, the unit circle $S^1 = \{z \in C : |z| = 1\} = \{e^{2\pi i\theta} : 0 \leq \theta \leq 1\}$ is a metric space w.r.t. the modified arc length distance d defined by $d(e^{2\pi i\theta_1}, e^{2\pi i\theta_2}) = |\theta_2 - \theta_1|$ $or\ [1 - |\theta_2 - \theta_1|]$ according as $|\theta_2 - \theta_1| \leq \frac{1}{2}$ or $|\theta_2 - \theta_1| > \frac{1}{2}$. S^1 is a compact metric space and there is a one-to-one correspondence between R/Z and S^1 via the map $\Psi: R/Z \to S^1$ given by $\Psi(x) = e^{2\pi ix}$.

Thus R/Z is identified with S^1 via the 1-1 correspondence $\Psi(x) = e^{2\pi ix}$. So, the map $f_m(x)$ on $R/Z \cong I/\sim$ can be defined on S^1 as $f_m(z = e^{2\pi ix}) = e^{2\pi i(mx)} = (e^{2\pi ix})^m = z^m$. This is a continuous map on S^1. Ultimately, $(S^1,\ f_m)$ is a topological dynamical system.

2.10: m-nary expansions and m-adic rationals:

It is well known that every real number x in the unit interval I = [0, 1] has a binary expansion given as $x = \sum_{i=1}^{\infty} \frac{x_i}{2^i}$ where $x_i \in \{0,1\}$ which is unique except for the dyadic rationals, rational numbers of the form $\frac{k}{2^n}$ where $0 < k < 2^n - 1$ and $n \in N$. In case of a dyadic rational, we have two different expansions, one ending with an infinite tail of 0's and the other with an infinite tail of 1's as the digits of expansion. We have also similar cases and notions for ternary expansions.

In a similar fashion for those of binary and ternary expansions and dyadic rationals, for any $m \in N\ (m > 3)$ and $x \in I = [0,1]$, we can talk about m-nary expansions. For every $x \in I$, we have an expansion $x = \sum_{i=1}^{\infty} \frac{x_i}{m^i}$, similar to those of the binary and ternary expansions in structure, where $x_i \in \{0,1,2,..., m-1\}$, which we call an m-nary expansion. Also, a rational number of the form $\frac{k}{m^n}$ in the unit interval $I = [0,1]$, where $0 < k < m^n - 1$ and $n \in N$, may be defined as an m-adic rational. One can easily verify that every m-adic rational has two different expansions, one ending with an infinite tail of 0's in the numerator and the other with an infinite tail of $(m-1)^{'s}$ in the numerator, just like dyadic rational numbers. e.g., if $x = \sum_{i=1}^{\infty} \frac{a_i}{m^i}, a_n \neq 0, a_k = 0$ for all k>n, i.e., $x = \sum_{i=1}^{n} \frac{a_i}{m^i},$ is an expansion for an m-adic rational, then, $\sum_{i=1}^{n-1} \frac{a_i}{m^i} + \frac{a_n - 1}{m^n} + \sum_{k=n+1}^{\infty} \frac{a_k}{m^k}, \quad a_k = m-1, \forall k > n$, is the other expansion for x.

3. The Main Results:

Theorem: 3.1[The Proximity Theorem]: Let $x, y \in \Sigma_m$. Then, $d(x, y) \leq \frac{1}{m^n}$ if and only if x and y agree up to n-digits [i.e. if $x = (x_1, x_2, x_3, ...),\ y = (y_1, y_2, y_3, ...)$ then $x_i = y_i$ for $i = 1,2,3,......, n$]

Proof: Let $x=(x_1,x_2,x_3,.....),\ y=(y_1,y_2,y_3,.....)\in\Sigma_m$ and x and y agree up to the n -digits.

Then, $x_i-y_i=0$ for $i=1,2,3,.....,n$ and therefore, we have, $\sum_{i=1}^{n}\frac{|x_i-y_i|}{m^i}=0$

So, $d(x,y)=\sum_{k\geq 1}\frac{|x_k-y_k|}{m^k}=\sum_{i=1}^{n}\frac{|x_i-y_i|}{m^i}+\sum_{k>n}\frac{|x_k-y_k|}{m^k}=0+\sum_{k>n}\frac{|x_k-y_k|}{m^k}=\sum_{k>n}\frac{|x_k-y_k|}{m^k}$

Now, $x=(x_1,x_2,x_3,....),y=(y_1,y_2,y_3,....)\in\Sigma_m\Rightarrow x_i,y_i\in\{0,1,2,3,....,m-1\},\quad\forall i\in N$

$$\Rightarrow|x_i-y_i|\leq m-1,\quad\forall i\in N$$

$$\Rightarrow\frac{|x_k-y_k|}{m^k}\leq\frac{m-1}{m^k},\ \forall k>n$$

$$\therefore d(x,y)=\sum_{k>n}\frac{|x_k-y_k|}{m^k}\leq\sum_{k=n+1}^{\infty}\frac{m-1}{m^k}=\frac{m-1}{m^{n+1}}.\sum_{r=0}^{\infty}\frac{1}{m^r}=\frac{m-1}{m^{n+1}}.\frac{1}{1-\frac{1}{m}}=\frac{1}{m^n}$$

Conversely, let, $d(x,y)<\frac{1}{m^n}$. We need to show that x and y agree up to n -digits. If possible, let, x disagrees to y at least at one digit that precedes the n^{th} -digit, say at i^{th} -digit where $1\leq i\leq n-1$ and agrees at all other digits up to n -digits. Then,

$$d(x,y)=\sum_{k\geq 1}\frac{|x_k-y_k|}{m^k}=\frac{|x_i-y_i|}{m^i}+\sum_{k=n+1}^{\infty}\frac{|x_k-y_k|}{m^k}\geq\frac{|x_i-y_i|}{m^i}\geq\frac{1}{m^i}>\frac{1}{m^n},[\because i\leq n-1<n]$$

This contradicts our assumption that $d(x,y)<\frac{1}{m^n}$. Further, if x disagrees to y at more than one digit for $1\leq i\leq n-1$, then also proceeding as above we get contradictions. So, it follows that x and y must agree up to n -digit. ∎

Theorem: 3. 2: The shift map $\sigma:\Sigma_m\to\Sigma_m$ is topologically transitive.

Proof: To establish that the shift map σ is topologically transitive, by proposition2.5, we need to show that for any two non-empty open sets U and V of Σ_m, there exists $n\in N$ such that $\sigma^n(U)\cap V\neq\phi$.

Let $x=(x_1,x_2,x_3,......)\in U$ *and* $y=(y_1,y_2,y_3,......)\in V$ be arbitrary (since U and V are non-empty open sets, so we always have such points).

Now, $x\in U$, $y\in V$ and U,V are open sets. So, ∃ open balls $B(x,r_1)\subseteq U$ and $B(y,r_2)\subseteq V$. If $r=\min\{r_1,r_2\}$, then $B(x,r)\subseteq U$ and $B(y,r)\subseteq V$. We choose $n\in N$ such that $\frac{1}{m^n}<r$. Consider the point $z=(x_1,x_2,x_3,.....,x_n,y_1,y_2,y_3,.....)\in\Sigma_m$ which agrees with x up to the n^{th} term. Therefore, by Proximity Theorem, we have

that $d(x,z) \leq \dfrac{1}{m^n} < r \Rightarrow z \in B(x,r) \subseteq U$ and consequently it follows that $\sigma^n(z) \in \sigma^n(U)$.

Also, $\sigma^n(z) = (y_1, y_2, y_3, \ldots\ldots) = y \in V,\ y = \sigma^n(z) \in \sigma^n(U) \Rightarrow y = \sigma^n(z) \in \sigma^n(U) \bigcap V$

So, it follows that $\sigma^n(U) \bigcap V \neq \phi$ and hence $\sigma : \Sigma_m \to \Sigma_m$ is topologically transitive.■

Theorem: 3.3: The shift map $\sigma : \Sigma_m \to \Sigma_m$ is topologically mixing.

Proof: Let U and V be any two non-empty open sets in Σ_m. We show that there exists a non-negative integer n_0 such that $\sigma^n(U) \bigcap V \neq \phi.,\ \forall n \geq n_0$.

Let $x = (x_1, x_2, x_3, \ldots\ldots) \in U$ *and* $y = (y_1, y_2, y_3, \ldots\ldots) \in V$ be arbitrary (since U and V are non-empty, so we must have such points in U and V). Then, since, $x \in U$, $y \in V$ and U , V are open sets in Σ_m , $\exists$ open balls $B(x,r_1)$, $B(y,r_2)$ such that $B(x,r_1) \subseteq U$ and $B(y,r_2) \subseteq V$. If $r = \min\{r_1, r_2\}$, then $B(x,r) \subseteq U$, $B(y,r) \subseteq V$ and we can choose $k \in N$ such that $\dfrac{1}{m^k} < r$. We then construct a sequence $\{ z_n \}$ of points in Σ_m with the help of k , x and y such that

$$z_1 = (x_1, x_2, x_3, x_4 \ldots\ldots x_k, y_1, y_2, y_3, y_4 \ldots\ldots\ldots)$$
$$z_2 = (x_1, x_2, x_3, x_4 \ldots\ldots x_k, a_1, y_1, y_2, y_3, y_4 \ldots\ldots)$$
$$z_3 = (x_1, x_2, x_3, x_4 \ldots\ldots x_k, a_1, a_2, y_1, y_2, y_3, y_4 \ldots\ldots)\ ,\ \ldots\ldots\ldots,$$
$$z_i = (x_1, x_2, x_3, \ldots., x_k, a_1, a_2, \ldots., a_{i-1}, y_1, y_2, y_3, \ldots.)\ ,\ i \geq 2,\ a_i^{'s} \in \{0,1,2,\ldots., m-1\}$$

Here, every $z_i, i \geq 2$, is constructed by using the finite word obtained by taking first $(i-1)$ consecutive symbols of a fixed sequence $a = (a_1, a_2, a_3, a_4, \ldots\ldots, a_{i-1}, \ldots..) \in \Sigma_m$ chosen arbitrarily. More precisely, the first k letters of z_i, for each $i \geq 2$, is the finite word $x_{[1,k]} = (x_1, x_2, x_3, x_4, \ldots\ldots\ldots, x_k)$ taken from $x \in U$ and then follows the word $a_{[1,i-1]} = (a_1, a_2, a_3, a_4, \ldots\ldots, a_{i-1})$ taken from a and at last the sequence representing y i.e. $z_i = (x_{[1,k]}, a_{[1,i-1]}, y)$. In this case we can also use a fixed letter from the alphabet set $\{0,1,2,\ldots., m-1\}$ repeating it for $(i-1)$ times rather than using $a_{[1,i-1]}$.

Now, by using the Proximity Theorem, we clearly have,

$d(x,z_i) \leq \dfrac{1}{m^k} < r$ [$\because x$ and z_i agree up to the k^{th}-digits], for all $i \in N$. So, $z_i \in B(x,r) \subseteq U$ and hence $\sigma^{k+i-1}(z_i) \in \sigma^{k+i-1}(B(x,r)) \subseteq \sigma^{k+i-1}(U)$ for all $i \in N$.

Also, $\sigma^{k+i-1}(z_i) = (y_1, y_2, y_3, \ldots..) \in V,\ \sigma^{k+i-1}(z_i) \in \sigma^{k+i-1}(U)$ imply that $\sigma^{k+i-1}(U) \bigcap V \neq \phi$, for all $i \geq 2$. Therefore, $\sigma^n(U) \bigcap V \neq \phi$, for all $n \geq k$.

Hence, the shift map $\sigma : \Sigma_m \to \Sigma_m$ is topologically mixing. ■

- We recall that a topological dynamical system $f : X \to X$ on a compact metric space (X,d) is said to exhibit sensitive dependence on initial conditions(shortly SDIC) if $\exists\ \delta>0$, called the sensitivity constant, such that for any $x \in X$ and for any neighbourhood $N(x)$ of x, $\exists$ a point $y \in N(x)$ and a non-negative integer n such that $d(f^n(x), f^n(y)) \geq \delta$.

Theorem:3.4: The shift map $\sigma : \Sigma_m \to \Sigma_m$ has sensitive dependence on initial conditions.

Proof: Let $x \in \Sigma_m$ be arbitrary and $N(x)$ be an arbitrary neighbourhood of x. Then, by definition of a neighbourhood, there exists a non-empty open set G such that $x \in G \subseteq N(x)$. Now, $x \in G$, G is open in $\Sigma_m \Rightarrow \exists$ an open ball $B(x,r)$ such that $B(x,r) \subseteq G \subseteq N(x)$. Let $y \in B(x,r) \subseteq G \subseteq N(x)$ such that $x \neq y$ and x is very close to y. This is always possible to have a very close point to x, because we can choose a $k \in N$ as large as we want satisfying $\frac{1}{m^k} < r$ and for this large $k \in N$ we can construct the point y in such a way that this agrees with x up to k-digits. Then $d(x,y) \leq \frac{1}{m^k} < r$ and hence for large value of k, x will be too close to y.

Let $d(x,y) = \varepsilon$. Then, since x is very close to y and ε is very small, so, depending on the value of $\varepsilon > 0$, $\exists$ a large and unique $n \in N$ s. t. $\frac{1}{m^{n+1}} < \varepsilon \leq \frac{1}{m^n}$. Consider $d(x,y) = \varepsilon \leq \frac{1}{m^n}$

Then, $d(x,y) \leq \frac{1}{m^n} \Rightarrow x$ and y agree up to the n^{th} digit

$\Rightarrow (n+1)^{th}$ digits of x and y are different

$\Rightarrow$ The first digit of $\sigma^n(x)$ and $\sigma^n(y)$ are different

$$\Rightarrow d(\sigma^n(x), \sigma^n(y)) = \sum_{i=1}^{\infty} \frac{|x_{n+i} - y_{n+i}|}{m^i} = \frac{|x_{n+1} - y_{n+1}|}{m} + \sum_{i=2}^{\infty} \frac{|x_{n+i} - y_{n+i}|}{m^i} \geq \frac{1}{m}$$

Here, from the above relation it is clear that $\frac{1}{m}$ plays the role of sensitivity constant δ.

Thus for every $x \in \Sigma_m$ and any neighbourhood $N(x)$ of x, $\exists\ y \in N(x)$ and $n > 0$ satisfying $d(\sigma^n(x), \sigma^n(y)) \geq \delta$ for $\delta = \frac{1}{m}$.

Hence the shift transformation $\sigma : \Sigma_m \to \Sigma_m$ has sensitive dependence on initial conditions. ■

Theorem:3.5: The set $P(\sigma)$, the set of all the periodic points of the shift map σ, is dense in Σ_m.

Proof: We first show that σ has $m^n - m$ periodic points of period-n in Σ_m for $n \geq 2$. It is to be noted that if a definite block of n-digits from the set $\{0,1,2,3,4,........, m-1\}$ repeats indefinitely, then it is a periodic point of σ of period-n in Σ_m. A block of n-digits can be formed with the m distinct digits 0, 1, 2, 3, ……, $m-1$ in m^n-ways. These blocks contains the m-blocks formed by the same digit (e.g. 000000..0, 11111…1, 222222…2 etc.) which are not periodic points of period-n. These are, in fact, periodic points of period-1 i.e. fixed points. So, we have only $(m^n - m)$ numbers of periodic points of period-n in Σ_m.

Consider an arbitrary point $x \in \Sigma_m$. We show that for any $\varepsilon > 0$, however small, there is a point $p \in P(\sigma)$ such that $d(x, p) < \varepsilon$. Let $x = (x_1, x_2, x_3, ...)$. For the fixed small $\varepsilon > 0$, we can always find a positive integer $n \in N$ such that $\frac{1}{m^n} < \varepsilon$.

Now, we construct a periodic point $p \in P(\sigma)$ of period $(n+1)$ such that

$p = (x_1, x_2, x_3, ..., x_n, y, x_1, x_2, x_3, ..., x_n, y, x_1, x_2, x_3, ..., x_n, y, ...)$

i.e. p is constructed by repeating the word $W = (x_1, x_2, x_3, ..., x_n, y)$ infinite number of times so that it agrees with the digits of x up to n-terms and disagrees at $(n+1)^{th}$ digit such that $x_{n+1} \neq y$ and $d(x, p) \leq \frac{1}{m^n} < \varepsilon$.

Thus, for every $x \in \Sigma_m$ and $\varepsilon > 0$, $\exists$ $p \in P(\sigma)$ such that $d(x, p) < \varepsilon$. That is, however small $\varepsilon > 0$ may be, for any $x \in \Sigma_m$ there is always a point $p \in P(\sigma)$ which is at a distance less than the arbitrarily small quantity $\varepsilon > 0$. Hence the set $P(\sigma)$ is dense.■

Theorem3.6: The shift map σ on Σ_m is *Devaney* as well as *Auslander-Yorke chaotic*.

Proof: We have already proved in the Theorems 3.2, 3.4 and 3.5 that(i) σ is topologically transitive, (ii) it has sensitive dependence on initial conditions and (iii) the set $P(\sigma)$ of all the periodic points of σ is dense in Σ_m. That is, σ satisfies all the requirements for *Devaney* as well as *Auslander-Yorke chaoticity*. So, it is *Devaney* as well as *Auslander-Yorke chaotic*. ■

Theorem3.7: The shift map σ on Σ_m is generically δ-chaotic with $\delta = diam(\Sigma_m) = 1$.

Proof: In Theorem3.3, we have established that the shift transformation σ on Σ_m is topologically mixing. Since by Proposition2.6, a continuous topologically mixing map on a compact metric space is also topologically weak mixing, so, the shift transformation σ being a continuous topologically mixing map on the compact metric space Σ_m is topologically weak mixing.

Also, by Proposition 2.7, a continuous topologically weak mixing map on a compact metric space X is generically δ-chaotic on X with $\delta = diam(X)$. So, it follows that the shift transformation σ on Σ_m is generically δ-chaotic with $\delta = diam(\Sigma_m) = 1$.■

- We remember that a dynamical system (X, f) is modified weakly chaotic dependence on initial conditions if for any $x \in X$ and every neighbourhood $N(X)$ of x, there are $y, z \in N(X)$ [in modified weakly case $y \neq x,\ z \neq x$] such that $(y,z) \in X^2$ is *Li-Yorke*.

Theorem: 3.8: The Topological Dynamical System (Σ_m, σ) has modified weakly chaotic dependence on initial conditions.

Proof: Let $x = (x_1, x_2, x_3, ...x_n, ...) \in \Sigma_m$ be any point and $N(x)$ be any neighbourhood of x. Then there exists an open set(open neighbourhood) U of Σ_m such that $x \in U \subseteq N(x)$.

Now, since $x \in U$ and U is an open set, so, there exists an open ball $B(x, r)$ with some radius $r > 0$ such that $B(x, r) \subseteq U \subseteq N(x)$. Then for this $r > 0$, we can choose a sufficiently large positive integer n such that $\frac{1}{m^n} < r$. We now find two points $y, z \in B(x, r) \subseteq U \subseteq N(x)$ with $y \neq x,\ z \neq x$ such that the pair $(y, z) \in \Sigma_m^2$ is *Li-Yorke*. Before this we define some terms and notations which will help us to simplify our proof.

By a word in Σ_m we mean a finite sequence of digits, called letters, from the set $\{0,1,2,3, ..., m-1\}$. Words are denoted by $A, B, C, ..., P, Q, R, ...$ etc. If the words A and B consist of p and n letters respectively such that $A = (a_1, a_2, a_3, ..., a_p)$ and $B = (b_1, b_2, b_3, ...,b_q)$, then by the symbol AB we mean the composite word $(a_1, a_2, ..., a_p, b_1, b_2, ..., b_q)$ which consists of $(p+q)$-number of letters. Using the letters in $x = (x_1, x_2, x_3, ...x_n, ...) \in \Sigma_m$, we now define the words $W(x, 3n)$, $W(x, 5n)$, $W(x, 7n)$, …….etc. as follows:

$$W(x,3n) = (x^*_{3n+1}, x^*_{3n+2}, ..., x^*_{4n}, x_{4n+1}, x_{4n+2}, ..., x_{5n}),$$

$$W(x,5n) = (x^*_{5n+1}, x^*_{5n+2}, ..., x^*_{6n}, x_{6n+1}, x_{6n+2}, ..., x_{7n}),$$

$$W(x,7n) = (x^*_{7n+1}, x^*_{7n+2}, ..., x^*_{8n}, x_{8n+1}, x_{8n+2}, ..., x_{9n}), \dots \text{ and so on.}$$

Note that each of the above words contains $2n$ letters, first n of which are the m-nary complements of the letters in the corresponding places of x and the rest n letters are just the letters in the corresponding places of x. In all the above words $x^*_k = (m-1) - x_k, \forall k$.

Now take $y = (x_1, x_2, ..., x_n, x^*_{n+1}, x^*_{n+2}, ..., x^*_{3n}, x_{3n+1}, x_{3n+2}, ..., x_{5n}, x_{5n+1}, x_{5n+2} ....)$

and $z = (x_1, x_2, ..., x_n, (0^*)^n, (0)^n, W(x,3n), W(x,5n), W(x,7n), W(x,9n), ...)$

where $(0^*)^n = \underbrace{0^*,0^*,0^*,....,0^*}_{n-terms}, (0)^n = \underbrace{0,0,0,.....0}_{n-terms}$ and $0^* = (m-1)-0 = m-1$

With these and the Proximity theorem for Σ_m, we now prove the theorem as follows:

Since y and z agree with x up to the n^{th} term, so, by Proximity theorem, we have, $d(x,y) \le \dfrac{1}{m^n} < r$, $d(x,z) \le \dfrac{1}{m^n} < r$ and consequently $y,z \in B(x,r) \subseteq U \subseteq N(x)$

Here, z contains infinitely many words of the type $W(x,(2k-1)n)$, where $k(\ge 2) \in N$, containing $2n$ letters each.

Also, $\sigma^{3n}(y) = (x_{3n+1}, x_{3n+2},......., x_{4n}, x_{4n+1}, x_{4n+2},........, x_{5n}, x_{5n+1}, x_{5n+2}......)$

$\sigma^{3n}(z) = (x^*_{3n+1}, x^*_{3n+2},......., x^*_{4n}, x_{4n+1}, x_{4n+2},........, x_{5n}, x^*_{5n+1}, x^*_{5n+2}......)$

$\sigma^{4n}(y) = (x_{4n+1}, x_{4n+2},......., x_{5n}, x_{5n+1}, x_{5n+2},........, x_{6n}, x_{6n+1}, x_{6n+2}......)$

$\sigma^{4n}(z) = (x_{4n+1}, x_{4n+2},......., x_{5n}, x^*_{5n+1}, x^*_{5n+2},........, x^*_{6n}, x_{6n+1}, x_{6n+2}........)$

Therefore, $\sup\limits_n d(\sigma^n(y),\sigma^n(z)) \ge d(\sigma^{3n}(y),\sigma^{3n}(z))$ and so

$$\underset{n\to\infty}{Lt} \sup_n d(\sigma^n(y),\sigma^n(z)) \ge \underset{n\to\infty}{Lt}\, d(\sigma^{3n}(y),\sigma^{3n}(z)) \ge \underset{n\to\infty}{Lt} \sum_{r=1}^{n} \frac{\left|x_{3n+r} - x^*_{3n+r}\right|}{m^r}$$

$$\ge \underset{n\to\infty}{Lt} \left\{ \frac{1}{m} + \frac{1}{m^2} + ... + \frac{1}{m^n} \right\} = \frac{1}{m-1}$$

Again,

$$0 \le \underset{n\to\infty}{Lt} \inf_n d(\sigma^n(y),\sigma^n(z))$$

$$\le \underset{n\to\infty}{Lt}\, d(\sigma^{4n}(y),\sigma^{4n}(z))$$

$$\le \underset{n\to\infty}{Lt}\, d(\ (x_{4n+1},.., x_{5n}, x_{5n+1},.., x_{6n}, x_{6n+1},..),(x_{4n+1},.., x_{5n}, x^*_{5n+1},.., x^*_{6n}, x_{6n+1},..))$$

$$\le \underset{n\to\infty}{Lt} \left\{ \left(\frac{m-1}{m^{n+1}} + \frac{m-1}{m^{n+2}} + ... + \frac{m-1}{m^{2n}} \right) + \left(\frac{m-1}{m^{3n+1}} + \frac{m-1}{m^{3n+2}} + ... + \frac{m-1}{m^{4n}} \right) + ... \right\}$$

$$= \underset{n\to\infty}{Lt} \left\{ \left(1 - \frac{1}{m^n} \right) . \frac{1}{m^n} \left(1 + \frac{1}{m^{2n}} + \frac{1}{m^{4n}} + \frac{1}{m^{6n}} .... \right) \right\}$$

$$= \underset{n\to\infty}{Lt} \left\{ \left(1 - \frac{1}{m^n} \right) . \frac{1}{m^n} . \frac{1}{1 - \frac{1}{m^{2n}}} \right\} = (1-0).0. \left(\frac{1}{1-0} \right) = 0$$

Now, $0 \le \underset{n\to\infty}{Lt} \inf\limits_n d(\sigma^n(y),\sigma^n(z)) \le 0 \Rightarrow \underset{n\to\infty}{Lt} \inf\limits_n d(\sigma^n(y),\sigma^n(z)) = 0$

Thus $\underset{n\to\infty}{Lt} \sup\limits_n d(\sigma^n(y),\sigma^n(z)) \ge \dfrac{1}{m-1}$ and $\underset{n\to\infty}{Lt} \inf\limits_n d(\sigma^n(y),\sigma^n(z)) = 0$

Hence, $(y,z)\in\Sigma_m^2$ is a *Li-Yorke* pair with modulus $\delta=\dfrac{1}{m-1}>0$ .Consequently, the dynamical system (Σ_m, σ) has modified weakly chaotic dependence on initial conditions. ■

- We recall that a dynamical system (X,f) is said to have chaotic dependence on initial conditions if for any $x\in X$ and every neighbourhood $N(X)$ of x, there is a $y\in N(X)$ such that the pair $(x,y)\in X^2$ is *Li-Yorke*.

Theorem:3.9: The dynamical system (Σ_m, σ) has chaotic dependence on initial conditions.

Proof: Let, $a=(a_1,a_2,a_3,......)\in\Sigma_m$ be an arbitrary point and $N(a)$ be any neighbourhood of a. Then there exists an open set(open neighbourhood) U of Σ_m such that $a\in U\subseteq N(a)$.
Now, since $a\in U$ and U is an open set, so, there exists an open ball $B(a,r)$ with some radius $r>0$ s. t. $B(a,r)\subseteq U\subseteq N(a)$. Then, for this $r>0$, we can choose a sufficiently large positive integer n such that $\dfrac{1}{m^n}<r$. We now find a point $b\in B(a,r)\subseteq U\subseteq N(a)$ such that the pair $(a,b)\in\Sigma_m^2$ is *Li-Yorke*. Here, also we use the similar terms and notations as in Theorem.3.8 to simplify our proof.
Using the letters in $a=(a_1,a_2,a_3,......a_n,......)\in\Sigma_m$, as in Theorem 3.8, we define the words $W(a,3n)$, $W(a,5n)$, $W(a,7n)$, ……. etc. as follows:

$$W(a,3n)=(a^*_{3n+1},a^*_{3n+2},................,a^*_{4n},a_{4n+1},a_{4n+2},..................,a_{5n}),$$
$$W(a,5n)=(a^*_{5n+1},a^*_{5n+2},................,a^*_{6n},a_{6n+1},a_{6n+2},..................,a_{7n}),$$
$$W(a,7n)=(a^*_{7n+1},a^*_{7n+2},................,a^*_{8n},a_{8n+1},a_{8n+2},.................,a_{9n}),\ \dots \text{ and so on.}$$

Now, using the above defined words we construct the point b as follows:

$$b=(a_1,a_2,........,a_n,(0^*)^n,(0)^n,W(a,3n),W(a,5n),W(a,7n),W(a,9n),..............)$$

where $(0^*)^n=\underbrace{0^*,0^*,0^*,.....,0^*}_{n-terms}$, $(0)^n=\underbrace{0,0,0,.....0}_{n-terms}$ and $0^*=(m-1)-0=m-1$

From the construction of b it is clear that b agrees with a up to the n^{th} term. So, by Proximity theorem, we have,

$$d(a,b)\le\frac{1}{m^n}<r \text{ and hence, } b\in B(a,r)\subseteq U\subseteq N(a)$$

Here, we see that b contains infinitely many words of the type $W(a,(2k-1)n)$, containing $2n$ letters each, where $k\ge 2$ is an integer.
Also, $\sigma^{3n}(b)=(a^*_{3n+1},a^*_{3n+2},.........,a^*_{4n},a_{4n+1},a_{4n+2},........,a_{5n},a^*_{5n+1},a^*_{5n+2},.......)$
And $\sigma^{4n}(b)=(a_{4n+1},a_{4n+2},.......,a_{5n},a^*_{5n+1},a^*_{5n+2},........,a^*_{6n},a_{6n+1},a_{6n+2}........)$
Therefore, $\sup_n d(\sigma^n(a),\sigma^n(b))\ge d(\sigma^{3n}(a),\sigma^{3n}(b))$ and so

$$\underset{n\to\infty}{Lt}\ \sup_n d(\sigma^n(a),\sigma^n(b)) \ge \underset{n\to\infty}{Lt}\ d(\sigma^{3n}(a),\sigma^{3n}(b)) \ge \underset{n\to\infty}{Lt} \sum_{r=1}^{n} \frac{\left|a_{3n+r}-a^*_{3n+r}\right|}{m^r}$$

$$\ge \underset{n\to\infty}{Lt}\left\{\frac{1}{m}+\frac{1}{m^2}+.....+\frac{1}{m^n}\right\} = \frac{1}{m-1}$$

Again, $0 \le \underset{n\to\infty}{Lt}\ \inf_n d(\sigma^n(a),\sigma^n(b))$

$$\le \underset{n\to\infty}{Lt}\ d(\sigma^{4n}(a),\sigma^{4n}(b))$$

$$= \underset{n\to\infty}{Lt}\ d((a_{4n+1},...,a_{5n},a_{5n+1},....,a_{6n},a_{6n+1},...),\ (a_{4n+1},...,a_{5n},a^*_{5n+1},...,a^*_{6n},a_{6n+1},....))$$

$$\le \underset{n\to\infty}{Lt}\left\{\left(\frac{m-1}{m^{n+1}}+\frac{m-1}{m^{n+2}}+...+\frac{m-1}{m^{2n}}\right)+\left(\frac{m-1}{m^{3n+1}}+\frac{m-1}{m^{3n+2}}+....+\frac{m-1}{m^{4n}}\right)+........\right\}$$

$$= \underset{n\to\infty}{Lt}\left\{\left(\frac{m-1}{m}+\frac{m-1}{m^2}+...+\frac{m-1}{m^n}\right).\left(\frac{1}{m^n}+\frac{1}{m^{3n}}+\frac{1}{m^{5n}}+..........\right)\right\}$$

$$= \underset{n\to\infty}{Lt}\left\{\left(1-\frac{1}{m^n}\right).\frac{1}{m^n}\left(1+\frac{1}{m^{2n}}+\frac{1}{m^{4n}}+\frac{1}{m^{6n}}.........\right)\right\}$$

$$= \underset{n\to\infty}{Lt}\left\{\left(1-\frac{1}{m^n}\right).\frac{1}{m^n}.\frac{1}{1-\frac{1}{m^{2n}}}\right\} = (1-0).0.\left(\frac{1}{1-0}\right) = 0$$

Now, $\quad 0 \le \underset{n\to\infty}{Lt}\ \inf_n d(\sigma^n(a),\sigma^n(b)) \le 0 \Rightarrow \underset{n\to\infty}{Lt}\ \inf_n d(\sigma^n(a),\sigma^n(b)) = 0$

So, it follows that $\underset{n\to\infty}{Lt}\ Sup_n\, d(\sigma^n(a),\sigma^n(b)) \ge \dfrac{1}{m-1}$ and $\underset{n\to\infty}{Lt}\ \inf_n d(\sigma^n(a),\sigma^n(b)) = 0$

Hence, $(a,b) \in \Sigma_m^2$ is a *Li-Yorke* pair with modulus $\delta = \dfrac{1}{m-1} > 0$. Therefore, the dynamical system (Σ_m, σ) has chaotic dependence on initial conditions. ■

4. Topological Conjugacy and Semi-Conjugacy:

Topological conjugacy between maps is a very powerful tool in the study of dynamical systems. This is due to the fact that most of the dynamical properties of a system are retained under topological conjugation. So, by studying the dynamical properties of a system, we can comment on the dynamical properties of other systems which are topologically conjugate to the first system.

In this section we discuss a little bit about this by defining the terms topological conjugacy and semi-conjugacy between two maps.

Let $f : X \to X$ and $g : Y \to Y$ be two continuous maps on the metric spaces X and Y. If there exists a homeomorphism $h : X \to Y$ such that $hof = goh$, then f is said to be *topologically conjugate* to the map g. In this case, *h* is called a *topological conjugacy*

[13].On the other hand, if for the continuous maps $f : X \to X$ and $g : Y \to Y$, there exists a surjection $h : X \to Y$ such that $hof = goh$, then f is *topologically semi-conjugate* to g.

The following results related to topological conjugacy are stated (without proof) as they are applied in some subsequent theorems.

Proposition 4.1: If the TDS $f : X \to X$ is topologically conjugate to the TDS $g : Y \to Y$ by the conjugacy map $h : X \to Y$, then,

(i) f is topologically transitive(resp., mixing/weakly mixing/exact/minimal) iff g is topologically transitive (resp., mixing/weakly mixing/exact/minimal).[13]

(ii) f is *DevC* (resp. *EDevC, MDevC, WMDevC*) if and only if g is *DevC* (resp. *EDevC, MDevC, WMDevC*)[13]

Proposition 4.2: If the TDS $f : X \to X$ is topologically semi-conjugated to the TDS $g : Y \to Y$,then, f is topologically transitive(resp., mixing) implies g is topologically transitive(resp., mixing).

Theorem: 4.3: The shift map $\sigma : \Sigma_m \to \Sigma_m$ and the map $f_m : R/Z \to R/Z$ such that $f_m(x) = mx$ (mod1) are topologically semi-conjugated.

Proof: Consider the map $\psi : \Sigma_m \to R/Z = I/\sim$ such that $\psi(x_1, x_2, x_3, ...) = \sum_{i=1}^{\infty} \frac{x_i}{m^i}$.

This map is well defined, because the series $\sum_{i=1}^{\infty} \frac{x_i}{m^i} \le \sum_{i=1}^{\infty} \frac{m-1}{m^i} = 1$ is convergent. We show that this mapping is a topological semi-conjugacy between $\sigma : \Sigma_m \to \Sigma_m$ and $f_m : R/Z \to R/Z$.

i) **ψ is surjective**: Since, every real number $x \in I = [0,1]$ has an m-nary expansion, so, we have, $x = \sum_{i=1}^{\infty} \frac{x_i}{m^i}$, where $x_i \in \{0,1,2,.....,m-1\}$. Then, the digits in the m-nary expansion for x will form the sequence $\bar{x} = (x_1, x_2, x_3, ...)$. As $x_i \in \{0,1,2,3,.........,m-1\}$, clearly, $\bar{x} \in \Sigma_m$. Also, by the definition of ψ, $\psi(\bar{x}) = \sum_{i=1}^{\infty} \frac{x_i}{m^i} = x$. Hence, ψ is surjective.

ii) **$\psi \circ \sigma = f_m \circ \psi$:**

For any $\bar{x} = (x_1, x_2, x_3, ...) \in \Sigma_m$, we have, $\sigma(\bar{x}) = (x_2, x_3, x_4, ...) \in \Sigma_m$ and

$$(\psi \circ \sigma)(\bar{x}) = \psi(\sigma(\bar{x})) = \psi(x_2, x_3, x_4, ...) = \sum_{i=1}^{\infty} \frac{x_{i+1}}{m^i}$$

Also, we have, $(f_m \circ \psi)(\bar{x}) = f_m(\psi(\bar{x})) = f_m(\sum_{i=1}^{\infty}\frac{x_i}{m^i}) = m(\sum_{i=1}^{\infty}\frac{x_i}{m^i})(\text{mod}1)$

$$= [x_1 + \sum_{i=1}^{\infty}\frac{x_{i+1}}{m^i}](\text{mod}1)$$

$$= \sum_{i=1}^{\infty}\frac{x_{i+1}}{m^i} = (\psi \circ \sigma)(\bar{x}),\ \forall \bar{x} \in \Sigma_m$$

Hence, we can conclude that $\psi \circ \sigma = f_m \circ \psi$.

Thus $\psi : \Sigma_m \to R/Z = I/\sim$ is a semi-conjugacy between σ and f_m. ■

Remark: Since, the pre-image of every m-adic rational number in $R/Z = I/\sim$ is a set of two distinct sequences in Σ_m, one with a tail of $0^{'s}$ and the other with a tail of $(m-1)^{'s}$, so, ψ is not injective. Hence, by restricting the domain of ψ, we make ψ injective and thereby make it a conjugacy between σ and f_m. For this we define the following space.

The Symbol Space $\Sigma_m/\sim$: The space $\Sigma_m/\sim$ is the symbol space with the equivalence relation '~' in Σ_m defined as follows:

$x \sim y$ iff $\exists k \in N$ s.t. $x_i = y_i, \forall i < k$ and $x_k = m-1, y_k = 0, x_i = 0, y_i = m-1,\ \forall i < k$

$where \quad x = (x_1, x_2, x_3, \ldots.., x_n, \ldots.), y = (y_1, y_2, y_3, \ldots.., y_n, \ldots.) \in \Sigma_m.$

We observe that the sequences identified as above are sequences with tails of 0's and (m-1)'s which correspond to two possible choices of m-nary digits for an m-adic rational number.

With these considerations in mind we have the following theorem:

Theorem: 4.4: The shift map $\sigma : \Sigma_m/\sim \to \Sigma_m/\sim$ and the map $f_m(x) = mx$ (mod1) on the space R/Z are conjugated by the mapping $\sigma : \Sigma_m/\sim \to R/Z$ defined by $\psi(x_1, x_2, x_3, x_4, \ldots\ldots.) = \sum_{i=1}^{\infty}\frac{x_i}{m^i}$.

Proof: With the same lines as in Theorem 4.3, we can establish that (i) ψ is surjective and (ii) $\psi \circ \sigma = f_m \circ \psi$.

Now, since, every m-adic rational number in [0, 1] is the image of two particular sequences in Σ_m which are equivalent in $\Sigma_m/\sim$, so, every m-adic rational has one and only one pre-image. Also, since, every non-m-adic rational has only one pre-image in $\Sigma_m/\sim$ ($\because$ every non-m-adic rational has unique m-nary expansion), so, it immediately follows that ψ is 1-1.

Therefore, ψ is a topological conjugacy between σ and f_m. ■

Theorem: 4.5: The map $f_m(x) = mx$ mod1 on *R/Z* is Devaney chaotic.

Proof: We have already established that the shift map $\sigma:\Sigma_m/\sim\,\to\Sigma_m/\sim$ is *Devaney chaotic* and is topologically conjugate to the map $f_m(x)=mx$ (mod1) on *R/Z*. Since, *Devaney chaoticity* retains under topological conjugacy, so, the map $f_m(x)=mx$ (mod1) on R/Z must be *Devaney chaotic*.

■

Theorem: 4.6: The shift map $\sigma:\Sigma_m\to\Sigma_m$ is *exact Devaney chaotic*, i.e. *EDevC*.

Proof: Let us first prove that the map $\sigma:\Sigma_m\to\Sigma_m$ is exact.
For this, let U be any non-empty open set in Σ_m. We now prove that there is an integer $k\in N$ such that $\sigma^k(U)=\Sigma_m$.
Since U is non-empty, so there exists at least one element $x\in U$. Again, since U is open in Σ_m, so, for $x\in U$, there must exists an open ball $B(x,r)$ such that $B(x,r)\subseteq U$.
Then we can choose some $k\in N$ such that $\frac{1}{m^k}\le r$. If we put $\frac{1}{m^k}=r_1$, then $r_1=\frac{1}{m^k}\le r$ and hence clearly $B(x,r_1)\subseteq B(x,r)\subseteq U$.
Then, for every $y\in B(x,r_1)$, we always have that $d(x,y)<r_1=\frac{1}{m^k}$.
From this it immediately follows that x and y agree at least up to the k^{th} term. Also after k^{th} term all the sequences in Σ_m may be tails of y. That is, $B(x,r_1)$ contains all the points whose first k digits agree with x and the tails are all the sequences of Σ_m. Hence the k^{th} iterates of all these points in $B(x,r_1)$ constitute the space Σ_m. i.e. $\sigma^k(B(x,r_1))=\Sigma_m$.

$$
\begin{aligned}
\text{Also, } B(x,r_1)\subseteq U &\Rightarrow \sigma^k(B(x,r_1))\subseteq\sigma^k(U)\\
&\Rightarrow \Sigma_m\subseteq\sigma^k(U)\\
&\Rightarrow \Sigma_m=\sigma^k(U)\quad,[\because\Sigma_m\supseteq\sigma^k(U)]
\end{aligned}
$$

Since, U is an arbitrary non-empty open set of Σ_m, so, the result $\Sigma_m=\sigma^k(U)$ is true for every non-empty open set U of Σ_m. Therefore, σ is an exact map on Σ_m.
In Theorem3.6, we have proved that σ is *Devaney chaotic*. So, it follows that σ is *exact Devaney chaotic* (*EDevC*). ■

Remark: As σ is *exact Devaney chaotic*(*EDevC*), therefore, it is also *MDevC* and *WMDevC*.

5.Conclusion:

In this paper we have extended some results of the shift transformation on Σ_2 to Σ_m and further we have proved some new results by applying the properties of topological conjugacy. To derive most of the results we have fruitfully applied the Proximity theorem and metric space properties. In Theorem3.8 and Theorem 3.9 we have proved that the shift map is modified weakly chaotic dependence on initial conditions and chaotic dependence on initial conditions respectively in a more explanatory way. Construction of *Li-Yorke* pairs have been done in these theorems in a clear-cut way. In Theorem: 4.4, it has been established that the shift map on $\Sigma_m/\sim$ is topologically conjugated to the map $f_m(x) = mx$ mod1 on R/Z and by retentivity of *Devaney chaos* under topological conjugation, we have concluded in Theorem: 4.5 that $f_m(x) = mx$ mod1 on R/Z is *Devaney chaotic* (*DevC*). Theorem: 4.6 establishes that the shift transformation is *exact Devaney chaotic*(*EDevC*) which is a more stronger condition on metric spaces as $EDevC => MDevC => WMDevC => DevC$. Most of the results are quite interesting and have profound applications in advanced analysis and discrete mathematics.

References:

1. Auslander J. and Yorke J.A., 1980.*Interval maps, factors of maps and chaos*. Tohoku Math. J. (32), 177-188.
2. Banks J. et al., 1992. *“ On Devaney’s definition of chaos”*, The American Mathematical Monthly, vol.99, no.4, pp.332-334.
3. Bhaumik I. and Choudhury B. S., 2009. *The Shift Map and the Symbolic Dynamics and Application of Topological Conjugacy*, Journal of Physical Sciences, vol.13, 149-160.
4. Blanchard F. et. al., 2002.*On Li-Yorke Pairs*, J. Reine Angew. Math., 547, 51-68.
5. Block L.S. and Copple W. A., 1992. *Dynamics in One Dimension*, Springer Lecture Notes, 1513, Springer Verlag, Berlin.
6. Denker M. et al., 1976. *Ergodic Theory on compact metric spaces*, Lecture Notes in Mathematics, 527, Springer-Verlag.
7. Devaney R.L., 1989. *An Introduction to Chaotic Dynamical Systems*, Second Edition, Addison-Wesley, Redwood City, California.
8. Du Bau-Sen, *On the Nature of Chaos*, ar Xiv: Math.DS 0602585.
9. Kitchens B. P., 1998.*Symbolic Dynamics-One Sided, Two Sided and Countable State Markov Shifts*, Universitext, Springer Verlag, Berlin.
10. Kwietniak D. and Misiurewicz M., 2005. “*Exact Devaney chaos and entropy*”, Qualitative Theory of Dynamical Systems, vol.6, no.1, pp.169-179.
11. Li T. Y. and Yorke J. A., 1975. “*Period three implies chaos*”, The American Mathematical Monthly, vol. 82, no.10, pp. 985-992, 1975.
12. Lind Douglas A. and Marcus Brian, 1995. *An Introduction to Symbolic Dynamics and Coding*, Cambridge University Press.
13. Lu T. et al., 2013. *The Retentivity of Chaos under Topological Conjugation*, http: //dx.doi.org/10.1155/2013/817831

14. Oprocha P., 2006. *Relations between distributional and Devaney Chaos*, Chaos, vol.16, no.3, Article ID 033112.
15. P. Touhey, 1997. *Yet another definition of chaos*, American Math. Monthly, 104.
16. Parry W., 1966. *Symbolic Dynamics and Transformation of the Unit Interval*, Trans. Amer. Math. Society, 122(2), April, 368-378.
17. Robinson C., 1999. *Dynamical System: Stability, Symbolic Dynamics and Chaos*, Second Edition, CRC Press, Boca Raton, FL.
18. Ruette S., 2003. *Chaos for continuous interval Map*., www.math.u-psud.fr/ruette/, Dec.15, 2003.
19. SchweizerB. andSmital J., 1994. *Measure of chaos and a spectral decomposition of dynamical systems of intervals*, Trans. Amer. Math. Soc., (344).
20. Shi Y. and Chen G., 2004. *Chaos of Discrete Dynamical System in Complete Metric Spaces*, Chaos, Solitons and Fractals, 22, pp.555-557.
21. SnohaL., 1992.*Dense chaos*, Comment. Math. Univ. Carolin.(33), pp.747-752.
22. Wang L. et al., 2007.*Distributional chaos in a sequence*. Nonlinear Analysis (67), pp. 2131-2136.