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Flat Curvature tensors on Hsu-Structure Manifold, and its application to relativity

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Abstract

The main object of this paper is to study the flatness of the Projectively flat, Conformally flat and Conharmonically flat Hsu-structure manifold. In last the application of flat curvature tensor in space time is given with an example.

Index Terms- Riemannian curvature Tensor, Projective curvature tensor, Conformal curvature tensor, Con-harmonic curvature tensor, Con-circular curvature tensor, Hsu-structure manifold.

1. INTRODUCTION

If on an even dimensional manifold V_n, n = 2m of differentiability class C^∞, there exists a vector valued real linear function ϕ, satisfying

$$\phi^2 = a^r I_n, \tag{1.1a}$$

$$\phi^2 X = a^r X \text{, for arbitrary vector field X.} \tag{1.1b}$$

where $\overline{X} = \phi X$, $0 \le r \le n$ and 'a' is a real or imaginary number.

Then { ϕ } is said to give to V_n a Hsu-structure defined by the equations (1.1) and the manifold V_n is called a Hsu-structure manifold.

Remark (1.1):The equation (1.1)a gives different structure for different values of 'a' and 'r'.

If $r = 0$, it is an almost product structure, if $a = 0$, it is an almost tangent structure, if $r = \pm 1$ and $a = +1$, it is an almost product structure, if $r = \pm 1$ and $a = -1$, it is an almost complex structure, if $r = 2$ then it is a GF-structure which includes π-structure for $a \neq 0$, an almost complex structure for $a = \pm i$,an almost product structure for $a = \pm 1$,an almost tangent structure for $a = 0$.

Let the Hsu-structure be endowed with a metric tensor g, such that

$g(\phi X,\phi Y)+a^r g(X,Y)=0$.

Then $\{\phi, g\}$ is said to give to V_n - metric Hsu-structure and V_n is called a metric Hsu-structure manifold.

Agreement(1.1): In what follows and the above, the equations containing X,Y,Z…….,etc.hold for these arbitrary vector in V_n.

The curvature tensor K, a vector -valued tri-linear function w.r.t. the connexion D is given by

$$R(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z, \tag{1.2a}$$

where

$$[X,Y] = D_X Y - D_Y X. \tag{1.2b}$$

$$R(\phi X,\phi Y)Z = R(X,Y)Z \tag{1.3}$$

The curvature tensor for the manifold of constant curvature H is given by

$$R(X,Y)Z = H\{g(Y,Z)X - g(X,Z)Y\}. \tag{1.4}$$

The Ricci tensor in V_n is given by

$$Ric(Y,Z) = (C_1^1 R)(Y,Z). \tag{1.5}$$

Where by $(C_1^1 R)(Y,Z)$, we mean the contraction of $R(X,Y)Z$ with respect to first slot.

For Ricci tensor, we also have

$$Ric(Y,Z) = Ric(Z,Y), \tag{1.6a}$$

$$Ric(Y,Z) = g(\gamma(Y),Z) = g(Y,\gamma(Z)), \tag{1.6b}$$

$$(C_1^1\gamma) = k \tag{1.6c}$$

Let W, C, L and V be the Projective, Conformal, Con-harmonic and Con-circular curvature tensors respectively given by

$$W(X,Y)Z = R(X,Y)Z - \frac{1}{(n-1)}[Ric(Y,Z)X - Ric(X,Z)Y]. \tag{1.7}$$

$$C(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}\{Ric(Y,Z)X - Ric(X,Z)Y - g(X,Z)\gamma(Y) + g(Y,Z)\gamma(X)\}$$
$$+\frac{k}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y]. \tag{1.8}$$

$$L(X,Y)Z = R(X,Y)Z - \frac{1}{(n-2)}[Ric(Y,Z)X - Ric(X,Z)Y - g(X,Z)\gamma(Y) + g(Y,Z)\gamma(X)]. \tag{1.9}$$

$$V(X,Y)Z = R(X,Y)Z - \frac{k}{n(n-1)}[g(Y,Z)X - g(X,Z)Y]. \tag{1.10}$$

Theorem (2.1): A Hsu-structure manifold $V_n\ (n \neq a^r - 1)$ of constant Riemannian curvature is flat.

Proof: From equation (1.4) we have the curvature tensor for the manifold of constant curvature tensor H is given by

$$R(X,Y)Z = H\{g(Y,Z)X - g(X,Z)Y\}. \tag{2.1}$$

Now applying ϕ on X and Y and using equation (1.3), we get

$$R(X,Y)Z = H\{\phi^*(Y,Z)\phi X - \phi^*(X,Z)\phi Y\}. \tag{2.2}$$

Comparing the equation (2.1) and (2.2) then either $H = 0$ or

$$\phi^*(Y,Z)\phi X - \phi^*(X,Z)\phi Y = g(Y,Z)X - g(X,Z)Y$$

Contracting of this equation w.r.t X, we get

$$\{n-(a^r-1)\}g(Y,Z)=0$$

Which is not possible, hence $H = 0$ and consequently R=0. This proves the statement.

Theorem (2.2): A projectively flat Hsu-structure manifold V_n, $n \neq 1-a^r$ is flat.

Proof: If the manifold is Projectively flat then from equaton (1.7), we get

$$(n-1)R(X,Y)Z = Ric(Y,Z)X - Ric(X,Z)Y \tag{2.3}$$

Now applying ϕ on X and Y in equation (2.3) and using equation (1.3), we get

$$(n-1)R(X,Y)Z = Ric(\phi Y,Z)\phi X - Ric(\phi X,Z)\phi Y \tag{2.4}$$

Comparing the equation (2.3) and (2.4) then we get

$$Ric(Y,Z)X - Ric(X,Z)Y = Ric(\phi Y,Z)\phi X - Ric(\phi X,Z)\phi Y \tag{2.5}$$

Contracting of equation (2.5) w.r.t X, we get

$$\{n-(1-a^r)\}Ric(Y,Z)=0$$

From theorem $n \neq 1-a^r$

Hence $Ric(Y,Z)=0$

Now putting $Ric = 0$ in equation (2.3), we get $R = 0$.

Hence the theorem.

Theorem (2.3): A Conformally flat Hsu-structure manifold V_n, $n^2 + n(a^r-3) - 2(a^r-1) \neq 0$ is flat.

Proof: If the manifold is conformally flat then from equation (1.8), we get

$$(n-2)R(X,Y)Z = \{Ric(Y,Z)X - Ric(X,Z)Y - g(X,Z)\gamma(Y) + g(Y,Z)\gamma(X)\}$$
$$+\frac{k}{(n-1)}[g(Y,Z)X - g(X,Z)Y]. \tag{2.6}$$

Now applying ϕ on X and Y in equation (2.6) and using equation (1.3), we get

$$(n-2)R(X,Y)Z = \{Ric(\phi Y,Z)\phi X - Ric(\phi X,Z)\phi Y - \phi^*(X,Z)\gamma(\phi Y) + \phi^*(Y,Z)\gamma(\phi X)\}$$
$$+\frac{k}{(n-1)}[\phi^*(Y,Z)\phi X - \phi^*(X,Z)\phi Y]. \tag{2.7}$$

Comparing the equation (2.6) and (2.7) then we get

$$Ric(Y,Z)X - Ric(X,Z)Y - g(X,Z)\gamma(Y) + g(Y,Z)\gamma(X) + \frac{k}{(n-1)}[g(Y,Z)X - g(X,Z)Y]$$
$$= Ric(\phi Y,Z)\phi X - Ric(\phi X,Z)\phi Y - \phi^*(X,Z)\gamma(\phi Y) + \phi^*(Y,Z)\gamma(\phi X) + \frac{k}{(n-1)}[\phi^*(Y,Z)\phi X - \phi^*(X,Z)\phi Y]$$

Contracting of this equation w.r.t X, we get

$$(n+2a^r-2)(n-1)Ric(Y,Z) - ka^r g(Y,Z) = 0 \tag{2.8a}$$
$$(n+2a^r-2)(n-1)\gamma Y - ka^r Y = 0 \tag{2.8b}$$

Contracting of this equation, we get

$\{n^2+n(a^r-3)-2(a^r-1)\}k\neq 0$

If $\{n^2+n(a^r-3)-2(a^r-1)\}\neq 0$, then $k=0$.

Putting $k=0$ in equation (2.8a), we get Ric=0, since $\{n^2+n(a^r-3)-2(a^r-1)\}\neq 0$. Putting $k=0$ and Ric=0 in equation (2.6), we get R=0, which proves the statement.

Theorem (2.4): If a Hsu-structure manifold V_n, $n=2(1-a^r)$ is Conformally flat, then its scalar curvature is 0 and it is Con-harmonically flat.

Proof: Putting $n=2-2a^r$ in equation (2.5a), we get $k=0$ which makes equation (2.6) in the form

$$(n-2)R(X,Y)Z=Ric(Y,Z)X-Ric(X,Z)Y-g(X,Z)\gamma(Y)+g(Y,Z)\gamma(X). \tag{2.9}$$

Using this equation in equation (1.9), we get the last part of the statement.

Theorem (2.5): If a Hsu-structure manifold V_n, $n\neq 1-a^r$ is Con-circularly flat, then it is flat.

Proof : If the manifold is Con-circular flat then from equation (1.10), we get

$$R(X,Y)Z=\frac{k}{n(n-1)}[g(Y,Z)X-g(X,Z)Y]. \tag{2.10}$$

Now applying ϕ on X and Y in equation (2.10) and using equation (1.3), we get

$$R(X,Y)Z=\frac{k}{n(n-1)}[\phi^*(Y,Z)\phi X-\phi^*(X,Z)\phi Y]. \tag{2.11}$$

Contracting of this equation w.r.t X, we get

$\{n+(a^r-1)\}g(Y,Z)=0$

Which is impossible, hence $k=0$. Putting $k=0$ in equation (2.10), we get R=0. Which proves the theorem.

Theorem (2.6): A Con-harmonically flat Hsu-structure manifold V_n, $n\neq 1-a^r$ is flat.

Proof: If the manifold is Con-harmonically flat then from equation (1.9), we get

$$(n-2)R(X,Y)Z=Ric(Y,Z)X-Ric(X,Z)Y-g(X,Z)\gamma(Y)+g(Y,Z)\gamma(X). \tag{2.12}$$

Applying ϕ on X and Y in (2.12), we get

$$(n-2)R(X,Y)Z=Ric(\phi Y,Z)\phi X-Ric(\phi X,Z)\phi Y-\phi^*(X,Z)\gamma(\phi Y)+\phi^*(Y,Z)\gamma(\phi X). \tag{2.13}$$

Comparing the equation (2.12) and (2.13) then we get

$$Ric(Y,Z)X-Ric(X,Z)Y-g(X,Z)\gamma(Y)+g(Y,Z)\gamma(X)=Ric(\phi Y,Z))$$
$$\phi X-Ric(\phi X,Z)\phi Y-\phi^*(X,Z)\gamma(\phi Y)+\phi^*(Y,Z)\gamma(\phi X$$

Contracting of this equation w.r.t X, we get

$$(n-2+2a^r)Ric(Y,Z)+kg(Y,Z)=0 \tag{2.14}$$

Again contracting the equation (2.14), we get

$\{n-(1-a^r)\}k=0$

If $n \neq 1-a^r$ then $k=0$. Putting $k=0$ in equation (2.14), we get Ric =0, hence R=0, this proves the theorem.

Application of Flat Curvature tensor in Spacetime:

A important feature of general relativity is the concept of a curved manifold. A useful way of measuring the curvature of a manifold is with an object called Riemannian curvature tensor. This tensor measures curvature by use of an affine connection by considering the effect of parallel transporting a vector between two curves. The similarity between the result of these two parallel transport routes essentially quantified by the Riemannian tensor. This property of the Riemannian tensor can be used to describe how initially parallel geodesic diverge. This is expressed by the equation of geodesic deviation and mean that the tidal force experienced in a gravitational field are a result of the curvature of spacetime.

Using the above procedure, the Riemannian tensor is defined as a type (1, 3) tensor and when fully written out explicitly contain the chirstoffel symbol and their first partial derivatives. The Riemannian tensor has 20 independent component. The vanishing of all these component over the region indicates that the spacetime is flat in that region. From the view point of geodesic deviation, this mean that initially parallel geodesic in that region of spacetime will stay parallel.

Example of application:

In Riemannian geometry, we consider the problem of finding an orthogonal coframe x^i, i.e a collection of 1-form forming a basis of cotangent space of every point with $\langle x^i, x^j \rangle = \delta^{ij}$ which are closed $(dx^i = 0, i = 1,2,3\ldots n)$. By the Poincare lemma, the x^i locally will have form df^i for some function f^i on the manifold, and thus provide an isometry of an open subset of V_n with an open subset of R^n. Such a manifold is called locally flat.

This problem reduces to a question on the coframe bundle of V_n. Suppose we have such a closed coframe

$\Theta = (x^1, \ldots\ldots\ldots\ldots, x^n)$,

If we have another coframe

$\Phi = (y^1, \ldots\ldots\ldots, y^n)$, then the two coframe would be related by an orthogonal transformation

$\Phi = V_n \Theta$, if the connection 1-form is ω, then we have

$$d\Phi = \omega \Lambda \Phi$$

On the other hand

$$d\Phi = (dV_n)\Lambda\Theta + V_n \Lambda d\Theta$$

$$d\Phi = dV_n \Lambda \Theta$$

$$d\Phi = (dV_n)V_n^{-1}\Lambda\Phi$$

But $\omega = (dV_n)V_n^{-1}$ is the Maurer-carton form for the orthogonal group. Therefore it obeys

the structural equation
$d\omega + \omega \Lambda \omega = 0$ and this is just the curvature of V_n ;
$\Omega = d\omega + \omega \Lambda \omega = 0$
After an application of the Frobenius theorem, we can conclude that a manifold V_n is locally flat iff its curvature tensor vanishes.

REFERENCES

[1] Mishra, R.S., (1973). A course in tensors with applications to Riemannian Geometry, *II edition Pothishala Pvt. Ltd., Allahabad.*

[2] Mishra, R.S. (1984). Structure on a differentiable manifold and their applications, *Chandrama Prakashan, 50-A, Balrampur House, Allahabad.*

[3] Habeeb M. Abood., (2010.) Almost Hermatian manifold with flat Bochner tensor, *European Journal of pure and applied mathematics*, 3(4), 730-736.

[4] Kirichenko V. F., (1977). Differential geometry of K-spaces, *Itogi, Nauki I Tehniki., VINITI, Moscow, problem geometri,* 8,139-161.

[5] Kobayashi S., and Nomizu K., (1969). Foundations of differential Geometry, V.II, *John Wiley and Sons.*

[6] De U.C. and Shaikh A.A., (2007). Diferential Geometry of Manifolds, *Narosa Pub. House, New Delhi.*

[7] Zhen, G. Cabrerizo, J. L., Fernandez, L.M. and Fernandez, M., (1997). On conformally flat at contact metric manifolds, *Indian J. Pure Appl. Math.,* 28, 725-734.

[8] Duggal, K.L., (1971). On differentiable structures defined by algebraic equations I, *Nijenhuis Tensor, N.S.,* 22, 238-242.

[9] Duggal, K.L., (1971). On differentiable structures defined by algebraic equations II, F-connexions, *Tensor, N.S.,* 22, 255-260.

[10] Eisenhart, L.P., (1966). Riemannian Geometry, *Princeton University Press*.

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