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On the Lattice of Convex Sets and Path Sets of a Connected Graph

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ABSTRACT

It is known that the set of all convex sets of a finite connected graph together with empty set partially ordered by set inclusion relation forms a lattice. Also the set of all path sets of a finite connected graph together with the empty set partially ordered by set inclusion relation forms a lattice.
In this paper we studied some of the properties of these lattices for complete bipartite graphs and cycles.

Key words: Convex sets, Connected graphs, Complete bipartite graph, lattice.

MSC subject classification: 06B99, 05C38.

1. Introduction

It is found that the lattice of normal subgroups of a group[1], the lattice of sublattices of a lattice[4], the lattice of convex sublattices of a lattice[5], [6], the lattice of subalgebras of a Boolean algebra[10], and the lattice of convex subgraphs of a directed graph[9]can be applied to study the internal structure of a group, lattice, Boolean algebra and directed graph.
Motivated from the above studies, the set of all convex sets of a finite connected graph together with the empty set is considered in [7] and it is found that this set also forms a lattice with respect to the partial order set inclusion. In this paper we studied properties of these lattices when the graph is complete bipartite and cycle. Also the set of all path sets of a connected graph is studied in [8]. In this paper we have shown that the lattice of path sets is complemented if the graph is a block with $|V(G)| \geq 3$. For terminologies and notations used in this paper we refer to [2] and [3].

2. Preliminaries

Let G be a finite connected graph. V(G) be the vertex set of G.A set C ⊆ V(G) is said to be convex in G if for every two vertices u, v ∈C, the vertex set of every u-v geodesic is contained in C. For a finite connected graph G, let the set of all convex sets in G together with empty set be denoted by Con(G).Define a binary relation ≤ on Con(G)by, for A, B∈Con(G), A≤B if and only if A ⊆ B, then clearly≤ is partial order on Con(G).Moreover $< \mathrm{Con(G)}, \leq >$form a lattice where for A, B∈Con(G), A∧B=A∩B and A∨B=<A ∪B>, where <A ∪B> is the convex set generated by A∪B or equivalently the smallest convex set containing A∪B.

For example, the lattice given in Fig2.2 represents the lattice $< \mathrm{Con(G)}, \leq >$of the connected graph G given in Fig 2.1

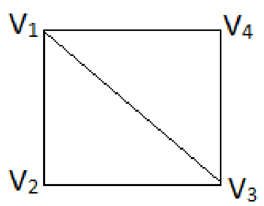

Fig 2.1

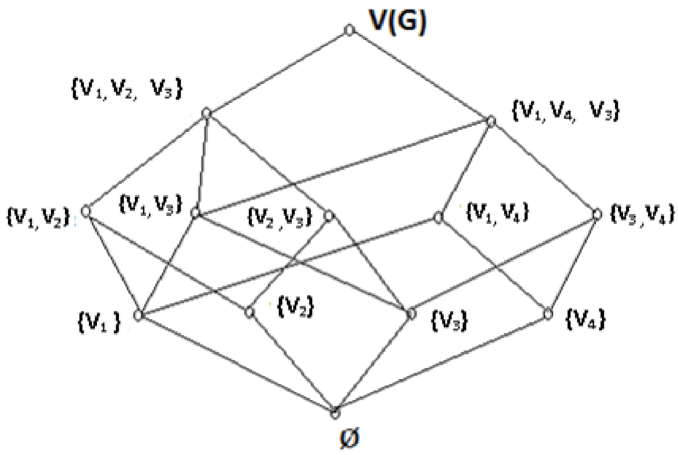

Fig 2.2

Throughout this paper we use Con(G) to represent the lattice $< \mathrm{Con(G)}, \leq >$.

3.On the lattice Con(G)

In [7], it has been shown that if a graph G is complete or cycle, then Con (G) is complemented. We extended this result to complete bipartite graphs.

Definition 3.1: A graph whose vertex set can be partitioned into two mutually disjoint nonempty subsets A and B such that each edge in G joins a vertex in A and a vertex in B, then G is called a bipartite graph. A bipartite graph is called complete bipartite if there is an edge between every vertex in A and every vertex in B.

Fig 3.2 represents the lattice Con (G) of the complete bipartite graph in Fig3.1

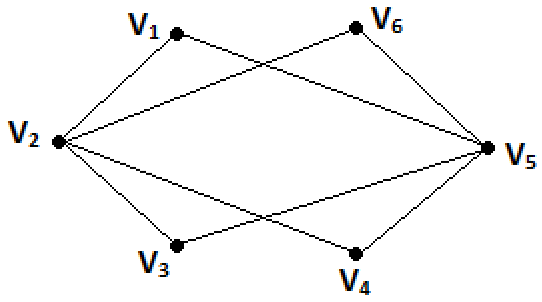

Fig 3.1

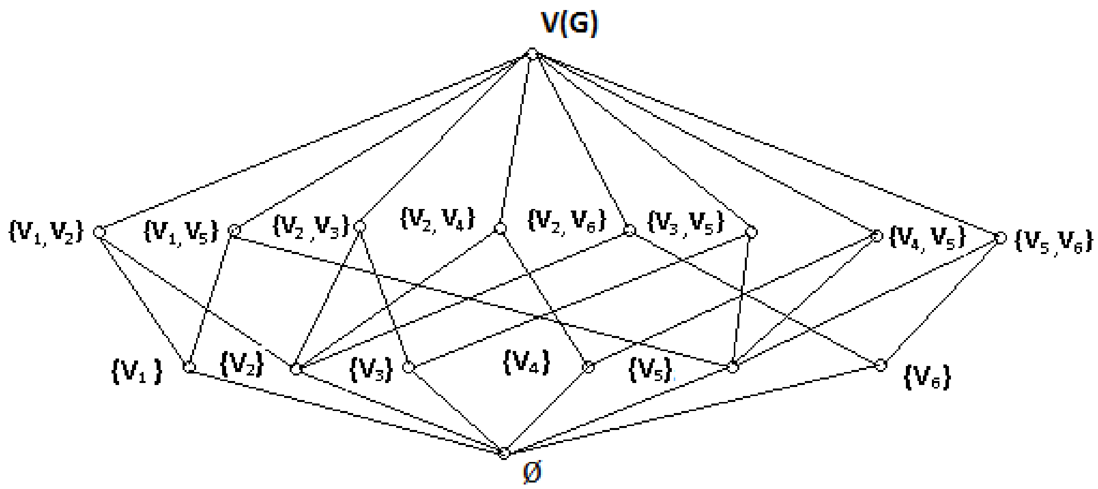

Fig 3.2

Theorem3.2: If graph G is a complete bipartite graph, then |Con(G)|= $(n+m+nm+2)$ where m and n are number of vertices in the partition of G.

Proof: Let G be a complete bipartite graph and V be its vertex set. A and B be partitions of V such that A∪B=V (G), A∩B=$\emptyset$ where A=$\{v_1, v_2, v_3, \dots, v_n\}$ B=$\{w_1, w_2, w_3, \dots, w_m\}$ with $n, m \geq 2$ and each edge of G has one end in A and one end in B, then convex sets of G will be of the form $\{v_i\}_{i=1,2,---n}, \{w_j\}_{j=1,2,---m}, \{v_i, w_j\}_{\substack{i=1,2,---n \\ j=1,2,---m}}$, so that total number of Convex sets will be$(n+m+nm)$ .The number of elements in Con(G) including $\emptyset$ and $V(G)$is $(n+m+nm+2)$.

Theorem3.3: If graph G is complete bipartite graph, then l(Con(G))=3.

Proof: Convex sets of a complete bipartite graph will be of the form $\{V_i\}, \{W_j\}, \{V_i W_j\}$where$\{V_i\} \in A, \{W_j\} \in B.$
Hence $\emptyset \prec \{V_i\}\ or\ \{W_j\} \prec \{V_{i,} W_j\} \prec V(G)$are maximal chain in Con(G) .Hence l(Con(G))=3 .

Theorem 3.4: If G is complete bipartite graph having partition with Cardinality ≥ 2, then Con (G) is complemented.

Proof: Let G be a complete bipartite graph and V be its vertex set. A and B be partitions of V such that A∪B=V (G), A∩B=∅ where A=$\{v_1, v_2, v_3, \ldots, v_n\}$ B=$\{w_1, w_2, w_3, \ldots, w_m\}$with $n, m \geq 2$. For the convex sets $\{v_i\}$, the complement will be $\{v_k\}$where $v_k \in$ A, $k \neq i$ and for the convex sets $\{w_j\}$, the complement will be $\{w_k\}$ where $w_k \in$ B and $k \neq j$.The complements of the convex sets $\{v_i, w_j\}$ will be $\{v_k\}$or $\{w_k\}$ where $k \neq i, j$. Hence Con (G) is complemented.

Theorem3.5: If a graph G is a cycle, then |Con (G) |=$\begin{cases} \frac{n^2}{2} + 2 \; if \; n \; is \; even \\ \frac{n(n+1)}{2} + 2 \; if \; n \; is \; odd \end{cases}$

and

l(Con(G))=$\begin{cases} \frac{n+2}{3} \; if \; n \; is \; even \\ \frac{n+3}{2} \; if \; n \; is \; odd \end{cases}$

Where n is number of vertices in G

Proof: Consider a graph G which is a cycle with n number of vertices, then it is of the form $v_1, v_2, v_3, \ldots, v_n, v_1$.

Case 1: If n is even, then convex sets will be of the form $\{v_1\}, \{v_1, v_2\}, \{v_1, v_2, v_3\} \ldots \left\{v_1, v_2, v_3, \ldots, v_{\frac{n}{2}}\right\}$ where v_1 can be renamed by $v_2, v_3, \ldots, v_n$ in cyclic order, so that the total number of convex sets will be $\left(n \times \frac{n}{2}\right) + 2 = \frac{n^2}{2}$. Hence the number of elements in Con (G) will be $\frac{n^2}{2} + 2$ including∅ and $V(G)$.The maximum chain in this lattice is $\emptyset \prec \{v_1\} \prec \{v_1, v_2\} \prec \{v_1, v_2, v_3\} \prec \cdots \prec \left\{v_1, v_2, v_3, \ldots, v_{\frac{n}{2}}\right\} \prec V(G)$ .So the length of the maximum chain is $\left(\frac{n}{2} + 2\right) - 1 = \frac{n+2}{2}$
.

Case 2: If n is odd, then convex sets will be of the form $\{v_1\}, \{v_1, v_2\}, \{v_1, v_2, v_3\} \ldots \left\{v_1, v_2, v_3, \ldots, v_{\frac{n+1}{2}}\right\}$ where v_1 can be renamed by $v_2, v_3, \ldots, v_n$ in cyclic order, so that the total number of convex sets will be $n \times \frac{(n+1)}{2}$. Hence the number of elements in Con(G) will be$\frac{n(n+1)}{2} + 2$ including ∅ and $V(G)$. The maximal chain in this lattice is $\emptyset \prec \{v_1\} \prec \{v_1, v_2\} \prec \{v_1, v_2, v_3\} \prec \cdots \prec \left\{v_1, v_2, v_3, \ldots, v_{\frac{n+1}{2}}\right\} \prec V(G)$ .So the length of the maximal chain is $\left(\frac{n+1}{2} + 2\right) - 1 = \frac{n+3}{2}$.

4. On the lattice PATH(G)

Let G be a connected graph and V(G) be the vertex set of G.A subset A of V(G) is said to be path set of G if for every u, v∈A the vertex set of all paths between u and v is contained in A. Let PATH(G)be the set of all paths sets of G together with the empty set

Define a binary relation on PATH (G) by for A, B∈PATH (G), A≤B if and only if A ⊆ B.Then PATH (G) forms a lattice with respect to this partial order where for A, B ∈ PATH (G), A∧B=A∩B and A∨B=<A ∪B>, where <A ∪B> is the smallest path set containing A∪B.

For example the lattice in Fig4.2 is the lattice PATH (G) of the graph G in Fig 4.1.

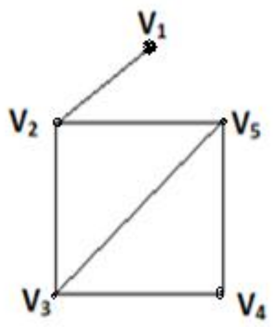

Fig 4.1

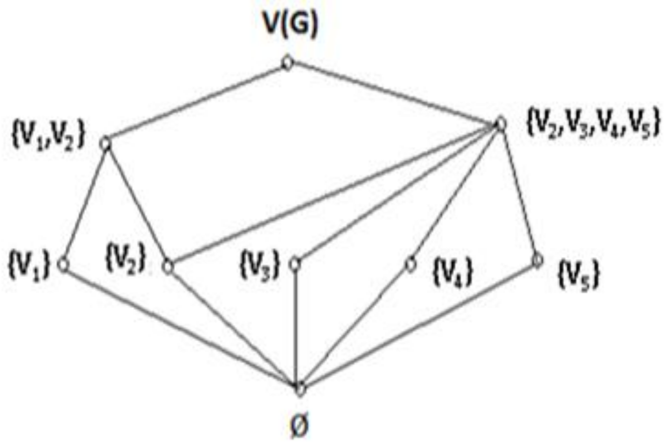

Fig 4.2

Theorem 4.1: In a graph G with $|V(G)| \geq 3,$ if G is a block then, PATH (G) is complemented

Proof: Claim: If G is a Block then PATH (G) is of the form shown in Fig4.3

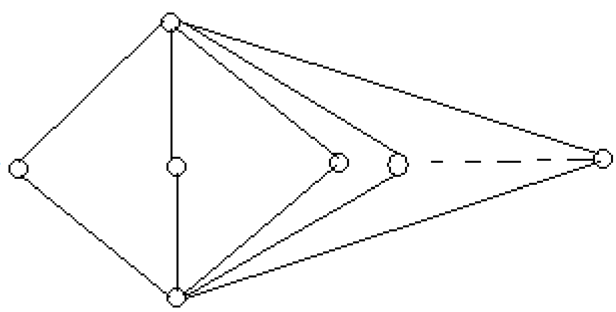

Fig 4.3

Proof of the claim: Let the graph G be a block and A ∈PATH (G) with $\emptyset \neq A$ ≠V(G). If $A \neq \{v_i\} i = 1, 2, 3, \_\_\_\_\_,$ then there exists a vertex $v_1 \in A$ which is incident with a vertex $v_2 \in V(G) - A$ since the graph is connected. Let v_3 be another vertex in A,

then v_3 cannot have a path with any element of $V(G) - A$ with out v_1 since A ∈PATH (G).Hence v_1 is a cut point of G, contradiction to G is a block.
From the above claim it is clear that if A ∈PATH (G), A $\neq \emptyset \neq V(G)$, then $A = \{v_i\}$ for some i so that the convex set $\{v_j\}$ for any $j \neq i$ will be the complement of A.

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