International Journal of Mathematics Research (IJMR).
ISSN 0976-5840 Volume 7, Number 2 (2015), pp. 203–207

http://www.ripublication.com/ijmr.htm

# Characterizations of a finite groups of order $pq$ and $p^2q$

**Kuwali Das**
*Bodoland University, Kokrajhar Assam.*

**Abstract**

In this paper, we study some interesting behavior of a finite group with order $pq$ where $p$ and $q$ are different primes. We find structure of the group of order $pq$ if the pair of primes $(p, q)$ are twin primes, cousin primes and sexy primes respectively. We also study the behavior of a group where order of the group is $p^2q$.

**AMS subject classification:** 20K01, 11A41.
**Keywords:** Abelian group, Cyclic group, Twin primes, Cousin primes, Sexy primes.

## 1. Introduction

A number $n > 1$ is prime if it has only two divisors 1 and itself. Twin prime was coined by Paul Stäckel $(1862 - 1919)$. Twin prime is a pair of prime numbers that has a prime gap of two, in other words, if $p$ and $p + 2$ are both prime numbers then $p$ and $p + 2$ are twin primes. Two is not considered as twin prime with the number three, since it violates the aforementioned rule. The first few twin primes are:

$$(3, 5),\ (5, 7),\ (11, 13),\ (17, 19),\ (29, 31),\ (41, 43),\ (59, 61),\ (71, 73),\ (101, 103), \ldots$$

Cousin primes are primes that differ by four. For example,

$$(3, 7),\ (7, 11)\ (13, 17),\ (19, 23),\ (37, 41), \ldots$$

Sexy primes are pair of primes of the form $(p, p + 6)$. That are named as Sex which is the Latin word for Six. The first few Sexy primes are:

$$(5, 11),\ (7, 13),\ (11, 17),\ (13, 19),\ (17, 23),\ (23, 29),\ (31, 37),$$

$$(37, 43),\ (41, 47),\ (47, 53), \ldots$$

A group is cyclic if it is generated by a single element. Also a group is Abelian if it satisfies the commutative law. It is well-known that every cyclic group is abelian.

In this paper, we study the structure of a group of order $pq$ or $p^2q$, where $p$ and $q$ are different primes. We can easily characterized the group by observing the nature of primes. Before going to main section, we mention the followings:

**Theorem 1.1.** **[2, Theorem 6.5]** Let $p$ and $q$ be two distinct prime numbers. To be definite let $p < q$. If $G$ is a group of order $pq$, then one of the following holds.

(I) $G$ is cyclic (Abelian).

(II) It has two generators $a$ and $b$ such that

$$a^p = e,\ b^q = e,\ a^{-1}ba = b^r,$$

for some $r$ such that $r \neq 1(mod\ \ q)$, $r^p \equiv 1(mod\ \ q)$ and $p$ divides $q - 1$ (Non-Abelian).

**Remark 1.2.** The above theorem gives that for any two primes $p < q$. Following cases happen.

(1) If $p$ does not divide $q - 1$, then any group $G$ of order $pq$ is cyclic.

(2) If $p$ divides $q - 1$ then there are only two non-isomorphic groups of order $pq$ one of which is commutative (which is again cyclic as $p$ and $q$ are different primes) other is non-commutative.

In [2, p 176], we have the following for a finite group of order $p^2q$; that is

A group of order $p^2q$, $p$ and $q$ being distinct prime numbers, is not simple. Further if $q < p$ and $q$ does not divide $p^2 - 1$ then every group of order $p^2q$ is Abelian.

In our next two sections, we find the following interesting results

**Theorem 1.3.** Let $G$ be a group and $o(G) = pq$ where $(p, q)$ is a pair of twin primes. Then $G$ is an Abelian group.

**Theorem 1.4.** Let $G$ be a group and $o(G) = pq$ where $(p, q)$ is a pair of cousin primes with $(p, q) \neq (3, 7)$. Then $G$ is an Abelian group.

**Theorem 1.5.** Let $G$ be a group and $o(G) = pq$ where $(p, q)$ is a pair of sexy primes with $(p, q) \neq (5, 11)$. Then $G$ is an Abelian group.

**Theorem 1.6.** Let $G$ be a group and $o(G) = pq$ where $(p, q)$ is a pair of primes where $q = p + 2k$, for some positive integer $k$ and $2k - 1$ is not a multiple of $p$. Then $G$ is an Abelian group.

**Theorem 1.7.** Let $G$ be a group and $o(G) = p^2q$ where $(q, p)$ is a pair of twin primes not equal to $(3, 5)$. Then $G$ is an Abelian group.

**Theorem 1.8.** Let $G$ be a group and $o(G) = p^2q$ where $(q, p)$ is a pair of cousin primes with $(q, p) \neq (3, 7)$. Then $G$ is an Abelian group.

**Theorem 1.9.** Let $G$ be a group and $o(G) = p^2q$ where $(q, p)$ is a pair of sexy primes with $(q, p) \neq (5, 11)$or $(7, 13)$. Then $G$ is an Abelian group.

**Theorem 1.10.** Let $G$ be a group and $o(G) = p^2q$ where $(q, p)$ is a pair of primes where $q = p + 2k$, for some positive integer $k$ and $4k^2 - 1$ is not a multiple of $q$ . Then $G$ is an Abelian group.

## 2. Characterization of a finite group of order $pq$, $p < q$

**Theorem 2.1.** Let $G$ be a group and $o(G) = pq$ where $(p, q)$ is a pair of twin primes. Then $G$ is an Abelian group.

*Proof.* Since $(p, q)$ is a pair of twin primes, therefore by definition of twin prime $q = p + 2$. Now,

$$\gcd(p, q - 1) = \gcd(p, p + 1) = 1,$$

since any two consecutive integers are always relatively prime. Therefore, $q - 1$ is not divisible by $p$.

By Remark of Theorem 6.5, [2][Theorem, 6.5, p 169], we can conclude that $G$ is cyclic and so Abelian. ■

**Theorem 2.2.** Let $G$ be a group and $o(G) = pq$ where $(p, q)$ is a pair of cousin primes with $(p, q) \neq (3, 7)$. Then $G$ is an Abelian group.

*Proof.* $(p, q) \neq (3, 7)$ is a pair of cousin primes, therefore $q = p+4$. If $\gcd(p, q-1) = \gcd(p, p + 3) = d$. then $d|p$ and $d|p + 3$ gives $d|3$ so $d = 1$ or 3.

Again, $d|p$ gives $d = 1$ or $p$. Now, if $d = 3$ then $p = d = 3$, but $p \neq 3$ therefore $d$ must be 1.

Hence, $d = \gcd(p, q-1) = 1$. Therefore, $q-1$ is not divisible by $p$. By Remark 1.2 of Theorem 6.5, [2][Theorem, 6.5, p 169], we can conclude that $G$ is cyclic and hence Abelian. ■

**Remark 2.3.** If $(p, q) = (3, 7)$ then by [2][Theorem, 6.5, p 169], we can conclude that $G$ is isomorphic to $Z_{21}$ or G is isomorphic to the group $\{a, b | a^3 = b^7 = e, a^{-1}ba = b^2\}$.

**Theorem 2.4.** Let $G$ be a group and $o(G) = pq$ where $(p, q)$ is a pair of sexy primes with $(p, q) \neq (5, 11)$. Then $G$ is an Abelian group.

*Proof.* $(p, q) \neq (5, 11)$ is a pair of sexy primes, therefore $q = p+6$. If $\gcd(p, q-1) = \gcd(p, p + 5) = d$. then $d|p$ and $d|p + 5$ gives $d|5$ so $d = 1$ or 5. Again, $d|p$ gives

$d = 1$ or $p$.

Now, if $d = 5$ then $p = d = 5$. but we are not considering here the case $p = 5$ so $d = \gcd(p, q-1) = 1$. Therefore, $q - 1$ is not divisible by $p$. By Remark 1.2 of Theorem 6.5, [2][Theorem, 6.5, p 169], we can conclude that $G$ is cyclic and so Abelian. ■

**Remark 2.5.** If $(p, q) = (5, 11)$ then by Remark 2.3 of Theorem 6.5, [2][Theorem, 6.5, p 169], we can conclude that $G$ is isomorphic to $Z_{55}$ or $\{a, b|a^5 = b^{11} = e, a^{-1}ba = b^3\}$. We can generalized the above result as follows

**Theorem 2.6.** Let $G$ be a group and $o(G) = pq$ where $(p, q)$ is a pair of primes where $q = p + 2k$, for some positive integer $k$ and $2k - 1$ is not a multiple of $p$. Then $G$ is an Abelian group.

*Proof.* $(p, q)$ is a pair of primes with $q = p + 2k$, for some positive integer $k$. Then if $\gcd(p, q-1) = \gcd(p, p+2k-1) = d$. then $d|p$ and $d|p+2k-1$ gives $d|2k-1$ so $d = 1$ or $2k - 1$. Again, $d|p$ gives $d = 1$ or $p$.

Now, if $d = 2k - 1$ then $p = d = 2k - 1$. but $2k - 1$ is not a multiple of $p$. $d = \gcd(p, q-1) = 1$. Therefore, $q - 1$ is not divisible by $p$. By Remark 1.2 of Theorem 6.5, [2][Theorem, 6.5, p 169], we can conclude that $G$ is cyclic and hence Abelian. ■

## 3. Characterization of a group of order $p^2q$, where $q < p$

**Theorem 3.1.** Let $G$ be a group and $o(G) = p^2q$ where $(q, p)$ is a pair of twin primes not equal to $(3, 5)$. Then $G$ is an Abelian group.

*Proof.* Since $(q, p$ is a pair of twin primes, therefore by definition of twin primes $p = q+2$. Since we have from [2][Problem 7, p 176], $G$ is abelian if $q$ does not divide $p^2-1$ In case twin primes, $p = q+2$, so, $p^2-1 = q^2+4q+3$. Now, $q|p^2-1$ if $q|3$. Hence, if $q = 3$ then $q|p^2-1$.

Therefore, $q$ does not divide $p^2 - 1$ if $(q, p) \neq (3, 5)$. Hence $G$ is Abelian if $(q, p) \neq (3, 5)$ is a pair of twin primes. ■

**Theorem 3.2.** Let $G$ be a group and $o(G) = p^2q$ where $(q, p)$ is a pair of cousin primes with $(q, p) \neq (3, 7)$. Then $G$ is an Abelian group.

*Proof.* $(p, q) \neq (3, 7)$ is a pair of cousin primes, therefore $q = p + 4$. As $p^2 - 1 = q^2 + 8q + 15$, so $q|p^2 - 1$ if $q|15$. There are only two possibilities for $q$, $q = 3$ or 5. But if $q = 5$ then $p = 9$, is not a prime. So,we can conclude that $q|p^2 - 1$ if $(q, p) = (3, 7)$. Hence $q$ does not divide $p^2-1$ if $(q, p) \neq (3, 7)$. Therefore in this case $G$ is Abelian. ■

**Theorem 3.3.** Let $G$ be a group and $o(G) = p^2q$ where $(q, p)$ is a pair of sexy primes with $(q, p) \neq (5, 11)$or $(7, 13)$. Then $G$ is an Abelian group.

*Proof.* Since $(q, p) \neq (5, 11)$or $(7, 13)$ is a pair of sexy primes, therefore $p = q + 6$. Here, $p^2 - 1 = q^2 + 12q + 35$ so $q|p^2 - 1$ if $q|35$. Hence there are only two possibilities for $q$, $q = 5$ or 7. So, we can conclude that $q|p^2 - 1$ if $(q, p) = (5, 11)$. or $(7, 13)$. Hence $q$ does not divide $p^2 - 1$ if $(q, p) \neq (5, 11)$. or $(7, 13)$. Therefore in this case $G$ is Abelian. ■

Now, we mention a generalization of the above results.

**Theorem 3.4.** Let $G$ be a group and $o(G) = p^2q$ where $(q, p)$ is a pair of primes where $p = q + 2k$, for some positive integer $k$ and $4k^2 - 1$ is not a multiple of $q$. Then $G$ is an Abelian group.

*Proof.* Here, $p^2 - 1 = q^2 + 8qk + 4k^2 - 1$. So, $p^2 - 1$ is not a multiple of $q$ if $4k^2 - 1$ is not multiple of $q$. So, in this case, we can conclude that $G$ is Abelian. ■

Some Example of the above result. In the above theorem if we put $k = 4$ then $p = q + 8$. Now, some example of pair of primes of this form are $(3, 11)$, $(5, 13)$, $(11, 19)$, $(23, 31)$, $(53, 61)$ ... Since, here, $4k^2 - 1 = 63$. So, in this case, the last theorem can be written as

**Theorem 3.5.** Let $G$ be a group and $o(G) = p^2q$ where $(q, p)$ is a pair of primes where $q = p + 8$, and $(q, p) \neq (3, 11)$ or $(5, 13)$. Then $G$ is an Abelian group.

Similarly if $k = 5$ then the pair of primes are $(3, 13)$, $(7, 17)$, $(13, 23)$, $(19, 29)$, $(31, 41)$, $(37, 47)$, ... we can write the (3.4) as

**Theorem 3.6.** Let $G$ be a group and $o(G) = p^2q$ where $(q, p)$ is a pair of primes where $q = p + 10$, and $(q, p) \neq (3, 13)$. Then $G$ is an Abelian group.

**Acknowledgement**

The author like to thank Professor Khanindra Chowdhury for his valuable suggestions.

## References

[1] Andersen, J.K. Gigantic Sexy and Cousin Primes. Post to primeform user forum. Nov., 3, 2005.

[2] Surjeet Singh and Qazi Zameeruddin, *Moden Algebra* Seventh Revised Edition, Vikas Publishing House.

[3] Hardy, G.H. and Wright, E.M., An introduction to the theoryof numbers, $5^{th}$ ed. Oxford, England, Clarendon Press 1979.

[4] Tomás Oliveira e Silva (7 April 2008). Tables of values of $pi(x)$ and of $pi^2(x)$. Aveiro University. Retrieved 7 January 2011.

[5] Weisstein, Eric W. Twin Primes. From MathWorld–A Wolfram We Resource. http://mathworld.wolfarm.com/Twinprimes.html.